Invariant Measures for Quantum Trajectories and Dark Subspaces

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Quantum trajectory is a sequence of random states $(\rho_n)_n$ modelling a quantum system on which an indirect measurement is repeatedly performed

$$\rho_0 \otimes \sigma$$

$\downarrow$ unitary interaction

$$U(\rho_0 \otimes \sigma)U^*$$

$\downarrow$ measure the probe with a PVM $(\pi_i)_i$

$$(\mathbb{I}_{sys} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^*(\mathbb{I}_{sys} \otimes \pi_i)$$

$\downarrow$ trace out the probe

With probability $\mathrm{Prob}(i) = \mathrm{tr}\left[(\mathbb{I}_{\mathrm{sys}} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^*\right]$
the state of the system becomes

$$\rho_1 = \mathrm{tr}_{\mathrm{probe}}\left[\frac{(\mathbb{I}_{\mathrm{sys}} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^*(\mathbb{I}_{\mathrm{sys}} \otimes \pi_i)}{\mathrm{Prob}(i)}\right]$$

$\downarrow$ take a new probe in state σ and repeat

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With probability $\mathrm{Prob}(i) = \mathrm{tr}\left[(\mathbb{I}_{\mathrm{sys}} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^*\right] = \mathrm{tr}(v_i \rho_0 v_i^*)$ the state of the system becomes

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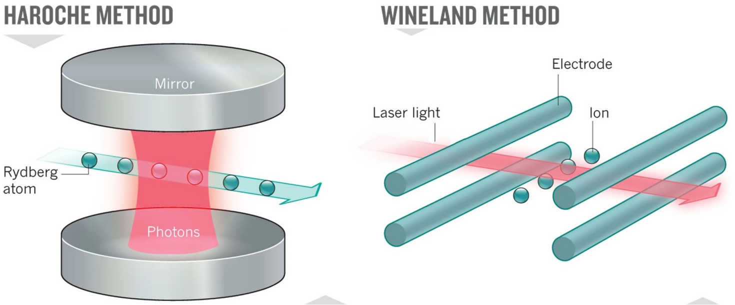
quantum trajectory $(\rho_n)_n$

- The goal is to **extract information** from a quantum system without any direct interaction of the system with a macroscopic apparatus.
- **Experimental implementation**: photons trapped in a cavity and probed with atoms. Allows to measure the number of photons without destroying them.
- **Serge Haroche** received for this experiment the 2012 Nobel prize in physics along with **David Wineland** (for a similar experiment)



Picture source: E.Hinds, R.Blatt, *Manipulating individual quantum systems.* Nature 492, 55 (2012)

Quantum trajectories

Quantum trajectory: sequence $(\rho_n)_n$ of random quantum states defined as

$$\rho_{n+1} = \frac{v_i \rho_n v_i^*}{\mathrm{tr}(v_i \rho_n v_i^*)} \quad \text{with prob.} \quad \mathrm{tr}(v_i \rho_n v_i^*)$$

where $v_1, \ldots, v_\ell \in \mathbb{C}^{d \times d}$ satisfy the stochasticity condition $\sum_{i=1}^{\ell} v_i^* v_i = \mathbb{I}$.

It is a **Markov chain** in the set of states $\mathcal{S}$.

What is its long-time behaviour? Invariant measures?

Everything holds for a measure μ on $M_d(\mathbb{C})$ satisfying some technical assumptions and the evolution

$$\rho_{n+1} = \frac{v \rho_n v^*}{\mathrm{tr}(v \rho_n v^*)} \quad \text{with prob.} \quad \mathrm{tr}(v \rho_n v^*)\, \mathrm{d}\mu(v)$$

Purification (and dark subspaces)

- Quantum trajectories preserve pure states:

$$x_{n+1} = \frac{v_i x_n}{\|v_i x_n\|} \quad \text{with probability} \quad \|v_i x_n\|^2 = \mathrm{tr}(v_i \, |x_n\rangle\langle x_n| \, v_i^*)$$

- **Purity may stay constant** along the trajectory:

if each Kraus operator is proportional to a unitary ($v_i = \lambda_i u_i$), then

$$\rho_{n+1} = \frac{v_i \rho_n v_i^*}{\mathrm{tr}(v_i \rho_n v_i^*)} = \frac{\lambda_i u_i \rho_n \overline{\lambda_i} u_i^*}{\mathrm{tr}(\lambda_i u_i \rho_n \overline{\lambda_i} u_i^*)} = u_i \rho_n u_i^*.$$

So ρ_{n+1} and ρ_n have the same eigenvalues $\Rightarrow$ purity stays the same

- Conditions for purification?

Theorem (Kümmerer & Maassen 2006)

Trajectories purify, ie. $\mathrm{dist}(\rho_n, \mathrm{Pure}) \xrightarrow{n\to\infty} 0$, iff there are no **dark subspaces**

Purification

Theorem (Benoist, Fraas, Pautrat, Pellegrini 2019)

If we assume that:

1. trajectories purify (i.e. no dark subspaces)
2. $\Phi(\rho) = \sum_i v_i \rho v_i^*$ is *irreducible* (i.e. it has a unique full-rank fixed point)

then there exists a **unique invariant probability measure** for $(\rho_n)_n$

Without irreducibility:
classification of invariant measures, based on decomposing Φ into irreducible parts

Proof: Construction of a sequence of states $(\widehat{\rho}_n)_n$ that depend only on the outcomes (and not on the initial state) and satisfy $\lim_{n\to\infty} d(\rho_n, \widehat{\rho}_n) = 0$

New aim: Classify the invariant measures for $(\rho_n)_n$ **without purification**
i.e. allowing dark subspaces (but keeping irreducibility)

Dark subspaces

A subspace $D \subset \mathbb{C}^d$ of dimension at least 2 is called a **dark subspace** if
$\forall\, n \in \mathbb{N} \quad \forall\, (i_1, \ldots, i_n) \quad \exists\, \lambda_{(i_1,\ldots,i_n)} \geq 0 \quad \exists\, U_{(i_1,\ldots,i_n)} \in \mathcal{U}(d)$:

$$v_{i_n} \cdots v_{i_1}\big|_D = \lambda_{(i_1,\ldots,i_n)} U_{(i_1,\ldots,i_n)}\big|_D$$

Equivalently, with π_D denoting the orthogonal projection on D:

$$\pi_D v_{i_1}^* \cdots v_{i_n}^* v_{i_n} \cdots v_{i_1} \pi_D = \lambda^2_{(i_1,\ldots,i_n)} \pi_D$$

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- 1-dim subspaces satisfy this trivially (but are not considered dark).
- in dim $d = 2$: $\mathbb{C}^2$ is dark iff all v_i's are proportional to unitaries.
- in general:
  - Hard to describe dark subspace for given matrices $v_1, \dots, v_\ell$
  - Dark subspaces can intersect non-trivially
  - There may be uncountably many of them (Ex. in Kümmerer & Maassen '06)

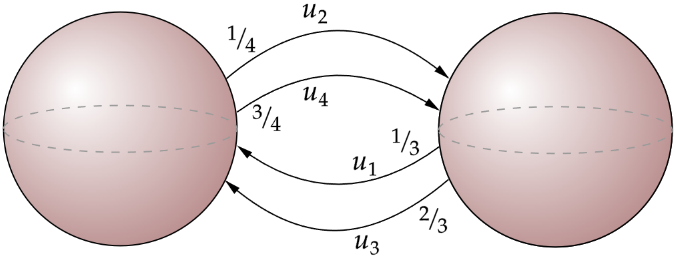
Example in dim $d = 4$

Let u_1, u_2, u_3, u_4 be 2×2 unitary matrices and consider the Kraus operators

$$v_1 = \begin{bmatrix} 0 & \sqrt{\frac{1}{3}}u_1 \\ \sqrt{\frac{1}{4}}u_2 & 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}}u_3 \\ \sqrt{\frac{3}{4}}u_4 & 0 \end{bmatrix}$$

There are two 2-dim dark subspaces:

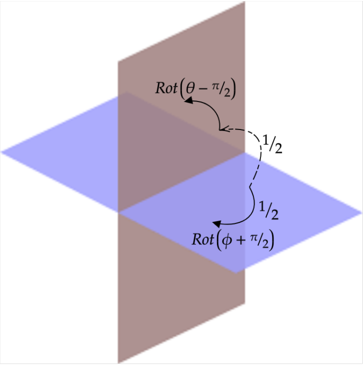
$$\{[z_1, z_2, 0, 0]^T : z_1, z_2 \in \mathbb{C}\} \quad \text{and} \quad \{[0, 0, z_3, z_4]^T : z_3, z_4 \in \mathbb{C}\}$$



Example of intersecting dark spaces

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\theta & \sin\theta & \cos\theta \\ \sin\theta & -\cos\theta & \sin\theta \\ 0 & 0 & 0 \end{bmatrix} \qquad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ \cos\phi & -\sin\phi & -\cos\phi \\ \sin\phi & \cos\phi & -\sin\phi \end{bmatrix}$$

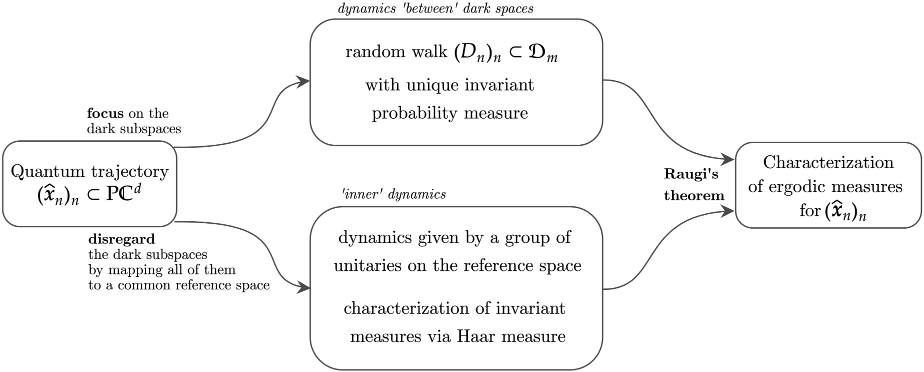
Again two dark subspaces: $\{[x, y, 0]^T : x, y \in \mathbb{C}\}$ and $\{[0, y, z]^T : y, z \in \mathbb{C}\}$



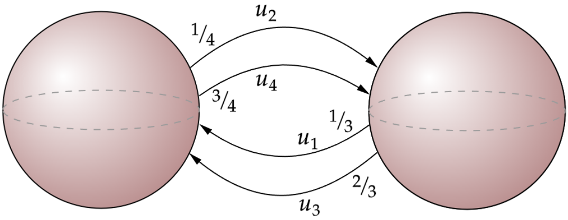
Strategy

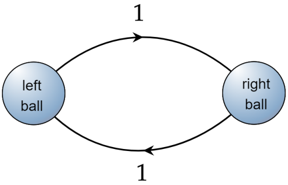
<u>Aim</u>: Classify invariant measures for $(\rho_n)_n$ **without purification** i.e. allowing dark subspaces (but keeping irreducibility)

Let's denote by $\mathcal{D}_m$ the set of *maximal dark subspaces*, i.e. those with the largest dimension, which we denote by r.



Zoom out: dynamics ‘between’ dark spaces





$$\chi_{\mathrm{inv}} = \frac{1}{2}\delta_{\mathrm{Left}} + \frac{1}{2}\delta_{\mathrm{Right}}$$

We can construct an invariant measure for $(\rho_n)_n$ as $\int_{\mathcal{D}_m} \mathsf{Unif}(\mathrm{P}(D))\, \mathrm{d}\chi_{\mathrm{inv}}(D)$

Zoom out: dynamics 'between' dark spaces

Theorem (Kümmerer, Maassen 2006)
Asymptotically, quantum trajectories perform a **random walk between dark subspaces** of the same dimension.

For $D \in \mathcal{D}_m$, with probability $\lambda_i^2 = \mathrm{tr}\big(v_i \frac{\pi_D}{\mathrm{tr}(\pi_D)} v_i^*\big)$ we have

$$D \longmapsto v_i D = \lambda_i U_i D$$

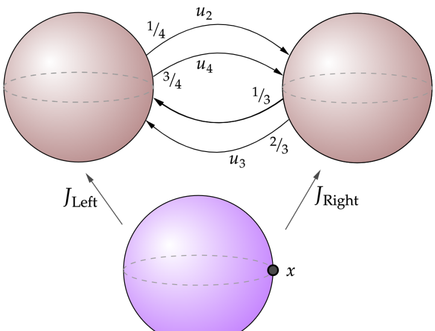
In terms of projectors:

$$\pi_D \longmapsto \frac{v_i \pi_D v_i^*}{\mathrm{tr}\big(v_i \frac{\pi_D}{\mathrm{tr}(\pi_D)} v_i^*\big)} = \pi_{v_i D}$$

Theorem (Benoist, Pellegrini, S.)
Φ is irreducible $\Rightarrow$ there exists a **unique invariant prob. meas.** for $(D_n)_n$.
Let's denote this measure by χ_{inv}.

Zoom in: the 'inner' dynamics. Reference space $\mathbb{C}^r$

- On $\mathrm{P}\mathbb{C}^r$ we consider **unitary operators** $\boldsymbol{u_{v_i,D} \propto J_{v_i D}^{-1} v_i J_D}$ induced by v_i's and a family of isometries $\{J_D \colon \mathbb{C}^r \to D\}_{D \in \mathcal{D}_m}$
- $G := \mathrm{cl}\langle\{u_{v_i,D} \colon i = 1 \ldots \ell, D \in \mathrm{supp}\, \chi_{\mathrm{inv}}\}\rangle$ carries the Haar measure. $\mathrm{Unif}[x]_G$ is invariant under the dynamics on $\mathrm{P}\mathbb{C}^r$ for any x.



Take $J_{\mathrm{Left}}, J_{\mathrm{Right}}$ as can. embeddings
$J_{\mathrm{Right}}^{-1} v_1 J_{\mathrm{Left}} \propto u_{v_1, D_{\mathrm{Left}}} = u_2 \colon \mathbb{C}^2 \to \mathbb{C}^2$
$G = \mathrm{cl}\,\langle\{u_1, u_2, u_3, u_4\}\rangle$

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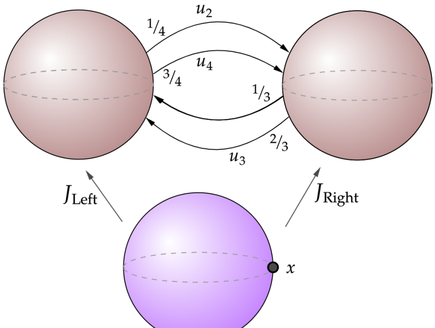
Now take χ_{inv} on $\mathcal{D}_m$ and consider $\chi_{\mathrm{inv}} \otimes \mathrm{Unif}[x]_G$ on $\mathcal{D}_m \times \mathrm{P}\mathbb{C}^r$. Send it to $\mathrm{P}\mathbb{C}^d$ via $\Psi \colon (D, z) \mapsto J_D z$.

We get an **invariant** measure $\nu_x = \Psi_*\big(\chi_{\mathrm{inv}} \otimes \mathrm{Unif}[x]_G\big)$ on $\mathrm{P}\mathbb{C}^d$



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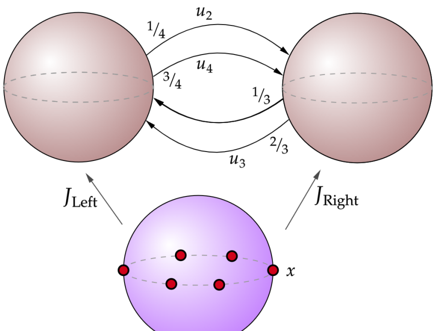

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$\mathrm{Unif}[x]_G = \sum_{x_i \in [x]_G} \frac{1}{6}\delta_{x_i}$

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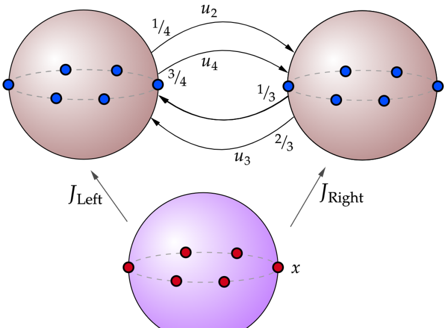

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$\mathrm{Unif}[x]_G = \sum_{x_i \in [x]_G} \frac{1}{6} \delta_{x_i}$

$$\nu_x = \overbrace{\frac{1}{2}}^{\chi_{\mathrm{inv}}(D_{\mathrm{Left}})} \sum_{y_i \in J_{\mathrm{Left}}[x]_G} \frac{1}{6} \delta_{y_i} + \overbrace{\frac{1}{2}}^{\chi_{\mathrm{inv}}(D_{\mathrm{Right}})} \sum_{y_i \in J_{\mathrm{Right}}[x]_G} \frac{1}{6} \delta_{y_i}$$

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- On $\mathrm{P}\mathbb{C}^r$ we consider (special) **unitary operators** $\boldsymbol{u_{v_i,D} \propto J_{v_i D}^{-1} v_i J_D}$ induced by v_i's and a family of isometries $\{J_D \colon \mathbb{C}^r \to D\}_{D \in \mathcal{D}_m}$
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Theorem: $(\nu_x)_{x \in \mathrm{P}\mathbb{C}^r}$ are the **only ergodic measures** iff $(J_D)_D$ is optimal

Definition: $(J_D)_D$ is called **optimal** if G is minimal (in terms of subgroups)

Proof is much easier if we assume that the dark subspaces intersect trivially (because $\Psi \colon \mathcal{D}_m \times \mathrm{P}\mathbb{C}^r \to \mathrm{P}\mathbb{C}^d$ is then invertible).

Without this assumption, we need an extra tool: Raugi's theorem

Main tool & main result

Let Π be the Markov kernel corresponding to (pure) quantum trajectories:

$$\Pi f(x) = \sum_i f\left(\frac{v_i x}{\|v_i x\|}\right)\|v_i x\|^2$$

Raugi's theorem (1992). If Π is equicontinuous, the map:

$$\{\text{ergodic measures}\} \ni \mu \mapsto \operatorname{supp}\mu \in \{\text{minimal sets}\}$$

is a **bijection**.

Theorem: $(\nu_x)_{x\in \mathrm{P}\mathbb{C}^r}$ are the **only ergodic measures** iff $(J_D)_D$ is optimal

Steps of the proof:

1. Π is indeed equicontinuous (Benoist, Hautecoeur, Pellegrini '24)
2. If $(J_D)_D$ is optimal, then $(\operatorname{supp}\nu_x)_{x\in \mathrm{P}\mathbb{C}^r}$ are the only minimal sets
3. So $(\nu_x)_{x\in \mathrm{P}\mathbb{C}^r}$ are the only ergodic measures (Raugi '92)

Summary

Theorem (T. Benoist, C. Pellegrini, AS)

If $\Phi(\rho) = \sum_i v_i \rho v_i^*$ is irreducible:

- All Π-ergodic measures are exactly the family $(\nu_x)_{x \in \mathrm{P}\mathbb{C}^r}$ constructed w.r.t. an optimal family.
- There is a **unique Π-invariant measure** iff an optimal family generates $\mathcal{SU}(r)$ or the symplectic group $\mathrm{Sp}(r/2)$ for the case $r \in 2\mathbb{N}$.

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1. T. Benoist, M. Fraas, Y. Pautrat, C. Pellegrini, *Invariant measure for quantum trajectories*, Probab. Theory Relat. Fields 174 (2019)
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3. H. Maassen, B. Kümmerer, *Purification of quantum trajectories*, IMS Lecture Notes-Monograph Series Dynamics & Stochastics 48 (2006)
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