Invariant Measures for Quantum Trajectories and Dark Subspaces

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Quantum trajectory is a sequence of random states $(\rho_n)_n$ modelling a quantum system on which an indirect measurement is repeatedly performed.

\[
\rho_0 \otimes \sigma \\
\downarrow \text{unitary interaction} \\
U(\rho_0 \otimes \sigma)U^* \\
\downarrow \text{measure the probe with a PVM } (\pi_i)_i \\
(\mathbb{I}_{\text{sys}} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^*(\mathbb{I}_{\text{sys}} \otimes \pi_i) \\
\downarrow \text{trace out the probe}
\]

With probability $\text{Prob}(i) = \text{tr} \left[ (\mathbb{I}_{\text{sys}} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^* \right]$ the state of the system becomes

\[
\rho_1 = \text{tr}_{\text{probe}} \left[ \frac{(\mathbb{I}_{\text{sys}} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^*(\mathbb{I}_{\text{sys}} \otimes \pi_i)}{\text{Prob}(i)} \right]
\]

\[
\downarrow \text{take a new probe in state } \sigma \text{ and repeat}
\]

quantum trajectory $(\rho_n)_n$
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With probability \(\text{Prob}(i) = \text{tr} \left[(\mathbb{I}_{sys} \otimes \pi_i)U(\rho_0 \otimes \sigma)U^*\right] = \text{tr}(v_i \rho_0 v_i^*)\)

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quantum trajectory \((\rho_n)_n\)
The goal is to **extract information** from a quantum system without any direct interaction of the system with a macroscopic apparatus.

**Experimental implementation**: photons trapped in a cavity and probed with atoms. Allows to measure the number of photons without destroying them.

**Serge Haroche** received for this experiment the 2012 Nobel prize in physics along with **David Wineland** (for a similar experiment)

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Quantum trajectories

**Quantum trajectory**: sequence \((\rho_n)_n\) of random quantum states defined as

\[
\rho_{n+1} = \frac{v_i \rho_n v_i^*}{\text{tr}(v_i \rho_n v_i^*)} \quad \text{with prob.} \quad \text{tr}(v_i \rho_n v_i^*)
\]

where \(v_1, \ldots, v_\ell \in \mathbb{C}^{d \times d}\) satisfy the stochasticity condition \(\sum_{i=1}^\ell v_i^* v_i = \mathbb{I}\).

It is a **Markov chain** in the set of states \(S\).

What is its long-time behaviour? Invariant measures?

Everything holds for a measure \(\mu\) on \(M_d(\mathbb{C})\) satisfying some technical assumptions and the evolution

\[
\rho_{n+1} = \frac{v \rho_n v^*}{\text{tr}(v \rho_n v^*)} \quad \text{with prob.} \quad \text{tr}(v \rho_n v^*) \, d\mu(v)
\]
Purification (and dark subspaces)

- Quantum trajectories preserve pure states:
  \[ x_{n+1} = \frac{v_i x_n}{\|v_i x_n\|} \text{ with probability } \|v_i x_n\|^2 = \text{tr}(v_i |x_n\rangle\langle x_n| v_i^*) \]

- **Purity may stay constant** along the trajectory:
  if each Kraus operator is proportional to a unitary \((v_i = \lambda_i u_i)\), then
  \[
  \rho_{n+1} = \frac{v_i \rho_n v_i^*}{\text{tr}(v_i \rho_n v_i^*)} = \frac{\lambda_i u_i \rho_n \lambda_i u_i^*}{\text{tr}(\lambda_i u_i \rho_n \lambda_i u_i^*)} = u_i \rho_n u_i^*.
  \]
  So \(\rho_{n+1}\) and \(\rho_n\) have the same eigenvalues \(\Rightarrow\) purity stays the same

- Conditions for purification?

**Theorem** (Kümmerer & Maassen 2006)

Trajectories purify, i.e. \(\text{dist}(\rho_n, \text{Pure}) \xrightarrow{n \to \infty} 0\), iff there are no **dark subspaces**
Theorem (Benoist, Fraas, Pautrat, Pellegrini 2019)

If we assume that:

1. trajectories purify (i.e. no dark subspaces)
2. $\Phi(\rho) = \sum_i v_i \rho v_i^*$ is irreducible (i.e. it has a unique full-rank fixed point)

then there exists a unique invariant probability measure for $(\rho_n)_n$

Without irreducibility:

classification of invariant measures, based on decomposing $\Phi$ into irreducible parts

Proof: Construction of a sequence of states $(\hat{\rho}_n)_n$ that depend only on the outcomes (and not on the initial state) and satisfy $\lim_{n \to \infty} d(\rho_n, \hat{\rho}_n) = 0$

New aim: Classify the invariant measures for $(\rho_n)_n$ without purification
i.e. allowing dark subspaces (but keeping irreducibility)
Dark subspaces

A subspace $D \subset \mathbb{C}^d$ of dimension at least 2 is called a **dark subspace** if

$$\forall n \in \mathbb{N} \quad \forall (i_1, \ldots, i_n) \quad \exists \lambda(i_1,\ldots,i_n) \geq 0 \quad \exists U(i_1,\ldots,i_n) \in \mathcal{U}(d) :$$

$$v_{i_n} \cdots v_{i_1} \big|_D = \lambda(i_1,\ldots,i_n) U(i_1,\ldots,i_n) \big|_D$$

Equivalently, with $\pi_D$ denoting the orthogonal projection on $D$:

$$\pi_D v_{i_1}^* \cdots v_{i_n}^* v_{i_n} \cdots v_{i_1} \pi_D = \lambda^2(i_1,\ldots,i_n) \pi_D$$

• in dim $d = 2$: $C_2$ is dark iff all $v$'s are proportional to unitaries.
• in general: Hard to describe dark subspace for given matrices $v_1, \ldots, v_\ell$. Dark subspaces can intersect non-trivially. There may be uncountably many of them (Ex. in Kümmerer & Maassen '06).
Dark subspaces

A subspace $D \subset \mathbb{C}^d$ of dimension at least 2 is called a **dark subspace** if

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\forall n \in \mathbb{N} \quad \forall (i_1, \ldots, i_n) \quad \exists \lambda_{(i_1,\ldots,i_n)} \geq 0 \quad \exists U_{(i_1,\ldots,i_n)} \in \mathcal{U}(d) : \\
v^*_{i_n} \cdots v^*_{i_1} \big|_D = \lambda_{(i_1,\ldots,i_n)} U_{(i_1,\ldots,i_n)} \big|_D
\]

Equivalently, with $\pi_D$ denoting the orthogonal projection on $D$:

\[
\pi_D v^*_{i_n} \cdots v^*_{i_1} v_{i_n} \cdots v_{i_1} \pi_D = \lambda^2_{(i_1,\ldots,i_n)} \pi_D
\]

- 1-dim subspaces satisfy this trivially (but are not considered dark).
- in dim $d = 2$: $\mathbb{C}^2$ is dark iff all $v_i$'s are proportional to unitaries.
- in general:
  - Hard to describe dark subspace for given matrices $v_1, \ldots, v_\ell$
  - Dark subspaces can intersect non-trivially
  - There may be uncountably many of them (Ex. in Kümmerer & Maassen '06)
Example in dim $d = 4$

Let $u_1, u_2, u_3, u_4$ be $2 \times 2$ unitary matrices and consider the Kraus operators

\[
v_1 = \begin{bmatrix} 0 & \sqrt{\frac{1}{3}} u_1 \\ \sqrt{\frac{1}{4}} u_2 & 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 & \sqrt{\frac{2}{3}} u_3 \\ \sqrt{\frac{3}{4}} u_4 & 0 \end{bmatrix}
\]

There are two 2-dim dark subspaces:

\[
\{ [z_1, z_2, 0, 0]^T : z_1, z_2 \in \mathbb{C} \} \quad \text{and} \quad \{ [0, 0, z_3, z_4]^T : z_3, z_4 \in \mathbb{C} \}
\]
Example of intersecting dark spaces

\[
v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \theta & \sin \theta & \cos \theta \\ \sin \theta & -\cos \theta & \sin \theta \\ 0 & 0 & 0 \end{bmatrix} \quad \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ \cos \phi & -\sin \phi & -\cos \phi \\ \sin \phi & \cos \phi & -\sin \phi \end{bmatrix}
\]

Again two dark subspaces: \{ \begin{bmatrix} x, y, 0 \end{bmatrix}^T : x, y \in \mathbb{C} \} and \{ \begin{bmatrix} 0, y, z \end{bmatrix}^T : y, z \in \mathbb{C} \}
**Strategy**

**Aim:** Classify invariant measures for \((\rho_n)_n\) **without purification** i.e. allowing dark subspaces (but keeping irreducibility)

Let’s denote by \(D_m\) the set of **maximal dark subspaces**, i.e. those with the largest dimension, which we denote by \(r\).
We can construct an invariant measure for \((\rho_n)_n\) as
\[
\int_{D_m} \text{Unif}(P(D)) \, d\chi_{\text{inv}}(D)
\]
Theorem (Kümmerer, Maassen 2006)
Asymptotically, quantum trajectories perform a random walk between dark subspaces of the same dimension.

For $D \in D_m$, with probability $\lambda_i^2 = \text{tr}(v_i \frac{\pi_D}{\text{tr}(\pi_D)} v_i^*)$ we have

$$D \mapsto v_i D = \lambda_i U_i D$$

In terms of projectors:

$$\pi_D \mapsto \frac{v_i \pi_D v_i^*}{\text{tr}(v_i \frac{\pi_D}{\text{tr}(\pi_D)} v_i^*)} = \pi v_i D$$

Theorem (Benoist, Pellegrini, S.)
$\Phi$ is irreducible $\Rightarrow$ there exists a unique invariant prob. meas. for $(D_n)_n$.
Let’s denote this measure by $\chi_{\text{inv}}$. 
On \( \mathbb{P} \mathbb{C}^r \) we consider unitary operators \( u_{v_i,D} \propto J_{v_i,D}^{-1} v_i J_D \) induced by \( v_i \)'s and a family of isometries \( \{ J_D : \mathbb{C}^r \to D \} \) \( D \in \mathcal{D}_m \).

\( G := \text{cl} \langle \{ u_{v_i,D} : i = 1 \ldots \ell, D \in \text{supp} \chi_{\text{inv}} \} \rangle \) carries the Haar measure. \( \text{Unif}[x]_G \) is invariant under the dynamics on \( \mathbb{P} \mathbb{C}^r \) for any \( x \).

Take \( J_{\text{Left}}, J_{\text{Right}} \) as can. embeddings

\[
J_{\text{Right}}^{-1} v_1 J_{\text{Left}} \propto u_{v_1,D_{\text{Left}}} = u_2 : \mathbb{C}^2 \to \mathbb{C}^2
\]

\( G = \text{cl} \langle \{ u_1, u_2, u_3, u_4 \} \rangle \)
Zoom in: the ‘inner’ dynamics. Reference space $\mathbb{C}^r$

- On $\mathbb{P}\mathbb{C}^r$ we consider **unitary operators** $u_{v_i,D} \propto J_{v_iD}^{-1} v_i J_D$ induced by $v_i$’s and a family of isometries $\{J_D : \mathbb{C}^r \to D\}_{D \in \mathcal{D}_m}$

- $G := \text{cl} \langle \{u_{v_i,D} : i = 1 \ldots \ell, D \in \text{supp} \chi_{\text{inv}}\} \rangle$ carries the Haar measure. $\text{Unif}[x]_G$ is invariant under the dynamics on $\mathbb{P}\mathbb{C}^r$ for any $x$.

Now take $\chi_{\text{inv}}$ on $\mathcal{D}_m$ and consider $\chi_{\text{inv}} \otimes \text{Unif}[x]_G$ on $\mathcal{D}_m \times \mathbb{P}\mathbb{C}^r$.
Send it to $\mathbb{P}\mathbb{C}^d$ via $\Psi : (D, z) \mapsto J_D z$.

We get an **invariant** measure $\nu_x = \Psi_* (\chi_{\text{inv}} \otimes \text{Unif}[x]_G)$ on $\mathbb{P}\mathbb{C}^d$

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\[
\text{Unif}[x]_G = \sum_{x_i \in [x]_G} \frac{1}{6} \delta_{x_i}
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On $\mathbb{P}\mathbb{C}^r$ we consider **unitary operators** $u_{v_i,D} \propto J_{v_iD}^{-1} v_i J_D$
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$\text{Unif}[x]_G = \sum_{x_i \in [x]_G} \frac{1}{6} \delta_{x_i}$

$\chi_{\text{inv}}(D_{\text{Left}}) \quad \chi_{\text{inv}}(D_{\text{Right}})$

$\nu_x = \frac{1}{2} \sum_{y_i \in J_{\text{Left}}[x]_G} \frac{1}{6} \delta_{y_i} + \frac{1}{2} \sum_{y_i \in J_{\text{Right}}[x]_G} \frac{1}{6} \delta_{y_i}$
On $\mathbb{P}\mathbb{C}^r$ we consider (special) **unitary operators** $u_{v_i,D} \propto J_{v_iD}^{-1}v_iJ_D$ induced by $v_i$'s and a family of isometries $\{J_D: \mathbb{C}^r \rightarrow D\}_{D \in \mathcal{D}_m}$

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**Theorem:** $(\nu_x)_{x \in \mathbb{P}\mathbb{C}^r}$ are the **only ergodic measures** iff $(J_D)_D$ is **optimal**

**Definition:** $(J_D)_D$ is called **optimal** if $G$ is minimal (in terms of subgroups)

Proof is much easier if we assume that the dark subspaces intersect trivially (because $\Psi: \mathcal{D}_m \times \mathbb{P}\mathbb{C}^r \rightarrow \mathbb{P}\mathbb{C}^d$ is then invertible).

Without this assumption, we need an extra tool: Raugi’s theorem
Main tool & main result

Let $\Pi$ be the Markov kernel corresponding to (pure) quantum trajectories:

$$\Pi f(x) = \sum_i f\left(\frac{v_i x}{\|v_i x\|}\right) \|v_i x\|^2$$

Raugi's theorem (1992). If $\Pi$ is equicontinuous, the map:

$$\{\text{ergodic measures}\} \ni \mu \mapsto \text{supp } \mu \in \{\text{minimal sets}\}$$

is a bijection.

Theorem: $(\nu_x)_{x \in \mathbb{P}\mathcal{C}^r}$ are the only ergodic measures iff $(J_D)_D$ is optimal

Steps of the proof:

1. $\Pi$ is indeed equicontinuous (Benoist, Hautecoeur, Pellegrini '24)
2. If $(J_D)_D$ is optimal, then $(\text{supp } \nu_x)_{x \in \mathbb{P}\mathcal{C}^r}$ are the only minimal sets
3. So $(\nu_x)_{x \in \mathbb{P}\mathcal{C}^r}$ are the only ergodic measures (Raugi '92)
Theorem (T. Benoist, C. Pellegrini, AS)

If $\Phi(\rho) = \sum_i \nu_i \rho \nu_i^*$ is irreducible:

- All $\Pi$-ergodic measures are exactly the family $(\nu_x)_{x \in P\mathbb{C}^r}$ constructed w.r.t. an optimal family.
- There is a **unique $\Pi$-invariant measure** iff an optimal family generates $SU(r)$ or the symplectic group $Sp(r/2)$ for the case $r \in 2\mathbb{N}$. 

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If $\Phi(\rho) = \sum_i v_i \rho v_i^*$ is irreducible:

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