A Max-Flow approach to Random Tensor Networks

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June 12, 2024

Esquisses Summer School 2

Introduction

Convention and notation

- Pure quantum state $|\psi\rangle$ in finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^D$.
- Mixed states are described by $\rho \in \mathcal{M}_D(\mathbb{C})$ such that $\rho \geq 0$ and $\operatorname{Tr} \rho = 1$, $\rho = |\psi\rangle\langle\psi|$ if ρ is pure.
- Bipartite pure state $|\psi_{AB}\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\psi_{AB}\rangle\langle\psi_{AB}| \to \rho_A = \operatorname{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|),$$

  where $\operatorname{Tr}_B(\cdot) := \mathrm{id}_A \otimes \operatorname{Tr}(\cdot)$.
- Rényi entropy is defined as:

$$S_n(\rho) := \frac{1}{1-n} \log(\operatorname{Tr}(\rho^n)).$$

- Von Neuman entropy is defined as:

$$S(\rho) := -\operatorname{Tr}(\rho \log \rho).$$

Random Tensor Network

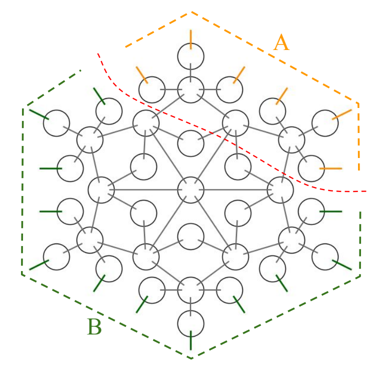


Figure 1: Generic (random) Tensor network $|\psi_G\rangle$ associated to G.

The main goal of this work:

$$|\psi_G\rangle \to \rho_A = \mathrm{Tr}_B\,|\psi_G\rangle\langle\psi_G| \to \lim_{D\to\infty} \mathbb{E}S_n(\rho_A) \sim \delta \log D + \text{“corrections”}.$$

where δ is the minimal cuts separating A from B.

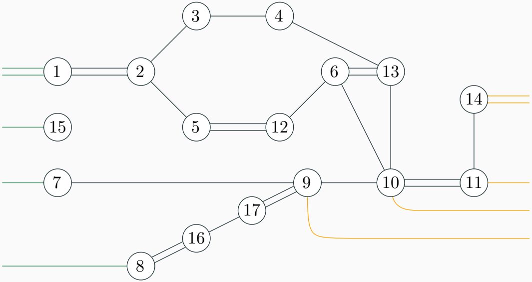
Random Tensor Network

Random Tensor Network

Definition

Let $G = (V, E)$ a connected undirected finite graph with edges E_b and half edges E_∂. Formally the set of edges and half edges are defined as follows

$$E_b := \{e_{x,y} \mid e_{x,y} = (x, y) : x, y \in V\},$$
$$E_\partial := \{e_x = (x, \cdot) : x \in V\},$$
$$E := E_b \sqcup E_\partial.$$



Random Tensor Network

Definition

From a given graph G we will define Hilbert space associated with each part of the graph.

- For each half-edge $e_x \in E_\partial$:

$$e_x \in E_\partial \to \mathcal{H}_{e_x} := \mathbb{C}^D.$$

- For each edges $e_{x,y} \in E_b$:

$$e_{x,y} \in E_b \to \mathcal{H}_{e_{x,y}} := \mathbb{C}^D \otimes \mathbb{C}^D \quad \text{and} \quad |\Omega_e\rangle := \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i_x, i_y\rangle .$$

- For each vertex $x \in V$:

$$x \in V \to \mathcal{H}_x := \bigotimes_{E \ni e \to x} \mathcal{H}_e \quad \text{and} \quad x \in V \to |g_x\rangle ,$$

  where $|g_x\rangle$ are random quantum state sampled from an i.i.d Gaussian distribution.

Random Tensor Network

Definition

A random tensor network $|\psi_G\rangle$ is defined as a projection of the vertex state over all the maximally entangled states $|\Omega_e\rangle$ for each $e_{x,y}$ in E_b where:

$$|\psi_G\rangle := \left\langle \bigotimes_{e \in E_b} \Omega_e \,\middle|\, \bigotimes_{x \in V} g_x \right\rangle \in \left(\mathbb{C}^D\right)^{\otimes |E_\partial|}.$$

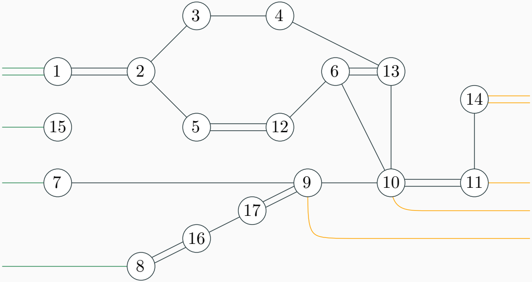


Figure 2: A tensor network depicting a tensor from $(\mathbb{C}^D)^{\otimes 10}$ obtained by contracting 17 tensors.

Moments

Let $A \subseteq E_\partial$ be a sub-boundary region of the graph G. We shall denote by $B := E_\partial \setminus A$ the complementary region of A.

The goal of this part is to compute the moments of ρ_A:

$$\rho_A \to \mathbb{E}\,\mathrm{Tr}(\rho_A^n),$$

where $\rho_A = \mathrm{Tr}_B |\psi_G\rangle\langle\psi_G|$ is the quantum state associated to the region A by tacking partial trace over the Hilbert space $\mathcal{H}_B$.

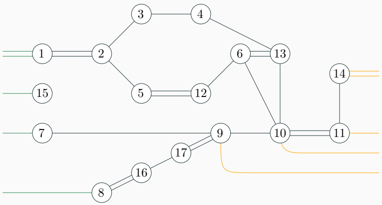


Figure 3: The boundary terms are partitioned into two subsets $B \sqcup A$.

Moments

Proposition

For any $A \subseteq E_\partial$, we have:

$$\forall n \in \mathbb{N}, \quad \mathbb{E}\left[\mathrm{Tr}(\rho_A^n)\right] = \sum_{\alpha=(\alpha_x)\in\mathcal{S}_n^{|V|}} D^{n|E_\partial| - H_G^{(n)}(\alpha)}$$

where $H_G^{(n)}(\alpha)$ is given given by:

$$H_G^{(n)}(\alpha) := \sum_{(x,\cdot)\in A} |{\gamma_x}^{-1}\alpha_x| + \sum_{(x,\cdot)\in B} |\,\mathrm{id}_x{}^{-1}\alpha_x| + \sum_{(x,y)\in E_b} |{\alpha_x}^{-1}\alpha_y|.$$

$\gamma_x := (n \cdots 1)$ is the total cycle in $\mathcal{S}_n$ and $|\alpha_x^{-1}\alpha_y|$ is the Cayley distance.

Example

For $n = 2$, the permutation group is only $\mathcal{S}_2$. Each permutation $\alpha_x \in \{\mathrm{id}_x, F_x\}$.

The (maximal) flow approach

Maximal flow

Definition

From the original graph G we construct a network $G_{A|B}$.

- The network $G_{A|B}$ is constructed by adding two extra vertices $\{\text{id}, \gamma\}$ and by connecting each of the boundary region A to γ and boundary region B to id.

Definition

- A flow in the network $G_{A|B}$ consist of all the possible paths that starts from id (source) and ends in γ (sink).
- A maximal flow is the maximal number of edge-disjoint paths from id to γ needed to remove such that the source (id) and the sink (γ) becomes completely disconnected, formally:

$$\mathrm{maxflow}(G_{A|B}) := \max\left\{\left|\mathcal{P}(G_{A|B})\right| \;:\; \text{paths in } \mathcal{P}(G_{A|B}) \text{ are edge-disjoint}\right\},$$

where

$$\mathcal{P}(G_{A|B}) := \{\pi_i : \quad \pi_i : \text{id} \to \gamma\},$$

Maximal flow

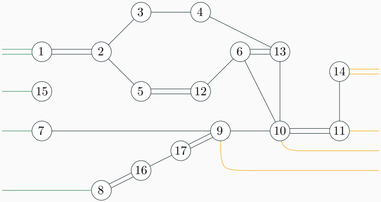


Figure 4: The region A and the region and its complementary region B where the boundary $B \sqcup A =: E_\partial$.

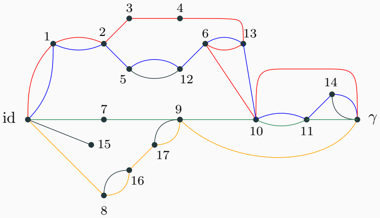


Figure 5: The maximum flow of the network $G_{A|B}$ is 4, the four augmenting paths achieving this value are colored.

Maximal Flow

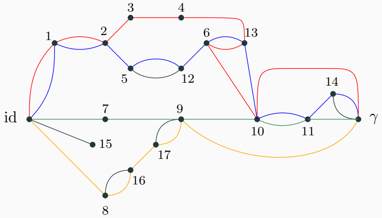


Figure 6: Network $G_{A|B}$ with coloured paths achieving the maximal flow.

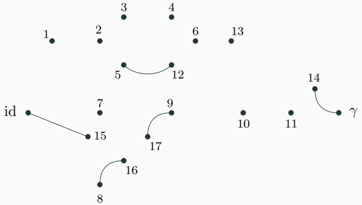


Figure 7: Residual network.

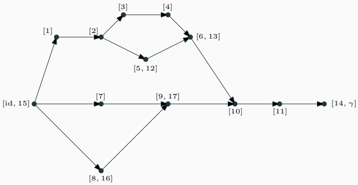


Figure 8: Ordered graph $G^{o}_{A|B}$.

Maximal flow

The maximal flow approach allows us to estimate the leading terms of $\mathbb{E}\left[\mathrm{Tr}(\rho_A^n)\right]$ as $D \to \infty$.

Theorem

For all $n \geq 1$, we have

$$\min_{\alpha \in \mathcal{S}_n^{|V|}} H^{(n)}_{G_{A|B}}(\alpha) = (n-1)\,\mathrm{maxflow}(G_{A|B}) = (n-1)\,\mathrm{maxflow}(G^o_{A|B}).$$

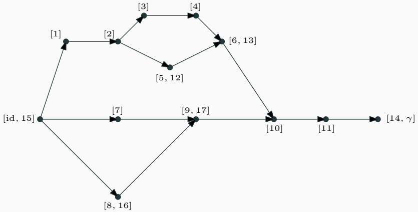


Figure 9: Ordered graph $G^o_{A|B}$.

Minimal cuts

Menger's Theorem asserts that the maximal flow in an oriented graph G is equal to the minimal number of edges to (remove) cut in order to make the source and the sink completely separated.

Theorem

Let G oriented graph:

$$\text{maxflow}(G) = \delta_G,$$

where δ_G is the minimal cut.

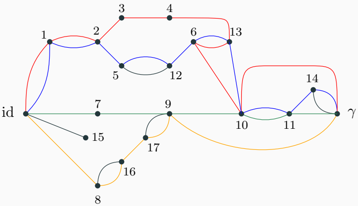


Figure 10: Maximal flow is 4 in the network $G_{A|B}$.

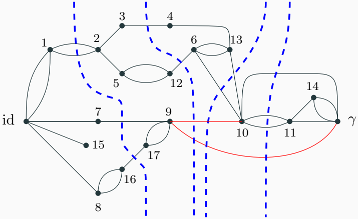


Figure 11: Minimal cut $\delta_{G_{A|B}} = 4$. In blue 4 cuts achieving $\delta_{G_{A|B}}$.

Main results

Series Parallel network

Definition

Let H_1 and H_2 two directed graph with there respective source s_i and sink t_i for $i \in \{1, 2\}$. A series-parallel network is a directed graph $G = (V, E)$ containing two distinct vertices $s \neq t \in V$, called the source and the sink that can be obtained recursively from the trivial network $G_{\text{triv}} = (\{s, t\}, \{\{s, t\}\})$ using the following two operations:

- Series concatenation: $G = H_1 \boxed{S} H_2$ is obtained by identifying the sink of H_1 with the source of H_2, i.e $t_1 = s_2$.
- Parallel concatenation: $G = H_1 \boxed{P} H_2$ obtained by identifying the source and the sink of H_1 and H_2, i.e. $s_1 = s_2$ and $t_1 = t_2$.

Graph and free probability

Definition

To a series-parallel graph G we associate a probability measure μ_G, defined recursively as follows.

- To the trivial graph $G_{\text{triv}} = (\{s,t\}, \{\{s,t\}\})$, we associate the Dirac mass at 1:

$$\mu_{G_{\text{triv}}} := \delta_1.$$

- Series concatenation, $G = H_1 \boxed{S} H_2$ we associate the measure:

$$\mu_G := \mu_{H_1} \boxtimes \Pi \boxtimes \mu_{H_2}.$$

- Parallel concatenation, $G = H_1 \boxed{P} H_2$ we associate the measure:

$$\mu_G := \mu_{H_1} \times \mu_{H_2}.$$

$\mathrm{d}\Pi := \frac{1}{2\pi}\sqrt{4t^{-1} - 1}\,\mathrm{d}t$ is the Marčhenko-Pastur distribution and $\boxtimes$ is the free convolution product.

Moment convergence

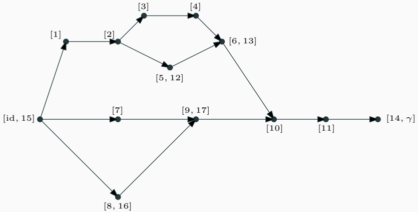


Figure 12: The obtained ordered graph $G^o_{A|B}$ is series parallel.

Theorem

Let $A \subseteq E_\partial$, assume the obtained ordered graph $G^o_{A|B}$ is series-parallel. The moment at large bond dimension of ρ_A are given by:

$$m^{(D)}_{n,A} := \frac{1}{D^{F(G^o_{A|B})}} \mathbb{E}\Big[\operatorname{Tr}\Big(\Big(D^{F(G^o_{A|B})} \rho_A \Big)^n \Big) \Big] \xrightarrow[D\to\infty]{} m_{n,G^o_{A|B}},$$

where $F(G^o_{A|B}) = \operatorname{maxflow}(G^o_{A|B})$ and $m_{n,G^o_{A|B}}$ are moments of a graph dependent measure $\mu_{G^o_{A|B}}$.

Example

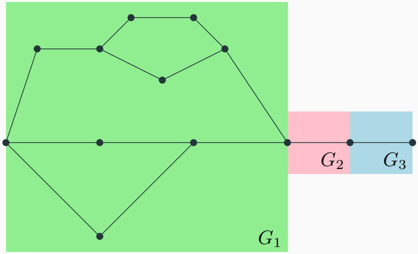


Figure 13: The ordered series-parallel graph $G^o_{A|B}$ factories as $G^o_{A|B} = G_1 \boxed{s} G_2 \boxed{s} G3$.

The associated measure

$$\mu_{G^o_{A|B}} = \mu_{G_1} \boxtimes \Pi \boxtimes \mu_{G_2} \boxtimes \Pi \boxtimes \mu_{G_3} = \mu_{G_1} \boxtimes \Pi^{\boxtimes 2}.$$

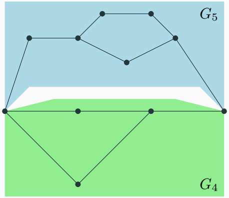


Figure 14: Graph $G_1 = G_4 \boxed{\mathsf{P}} G_5$ factorizes to parallel composition of G_4 and G_5. The associated measure $\mu_{G_1} = \mu_{G_4} \times \mu_{G_5}$.

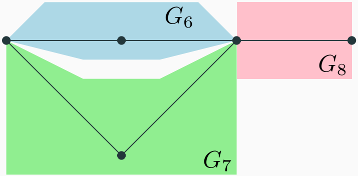


Figure 15: $G_4 = \left(G_6 \boxed{\mathsf{P}} G_7\right) \boxed{\mathsf{S}} G_8$ where $\mu_{G_4} = (\mu_{G_6} \times \mu_{G_7}) \boxtimes \Pi \boxtimes \mu_{G_8}$ and $\mu_{G_6} = \mu_{G_7} = \Pi$, while $\mu_{G_8} = \delta_1$.

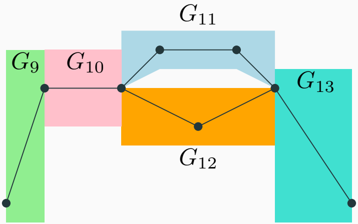


Figure 16: $G_5 = G_9 \boxed{\mathsf{S}} G_{10} \boxed{\mathsf{S}} \left(G_{11} \boxed{\mathsf{P}} G_{12}\right) \boxed{\mathsf{S}} G_{13}$ where $\mu_{G_5} = \Pi^{\boxtimes 3} \boxtimes \left(\Pi^{\boxtimes 2} \times \Pi\right)$

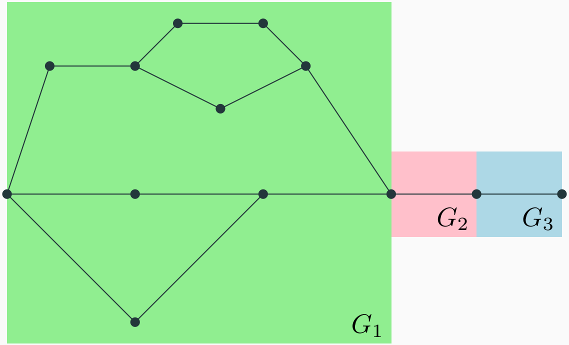


Figure 17: The series-parallel ordered graph $G^o_{A|B}$.

In the example considered here, we have:

$$\lim_{D\to\infty} \mathbb{E} D^{-4} \operatorname{Tr}\left[(D^4\rho_A)^n\right] = m_{n,G^o_{A|B}} = \int x^n \,\mathrm{d}\mu_{G^o_{A|B}}. \qquad (1)$$

where

$$\mu_{G^o_{A|B}} = \left\{\left[\Pi^{\boxtimes 3} \boxtimes (\Pi^{\boxtimes 2} \times \Pi)\right] \times [(\Pi \times \Pi) \boxtimes \Pi]\right\} \boxtimes \Pi^{\boxtimes 2}.$$

Main theorem

Theorem

Let boundary region $A \subseteq E_\partial$ in G, and let ρ_A the associated reduced state. Assuming the obtained ordered graph $G^o_{A|B}$ is a series-parallel one. Then the averaged Rényi and von Neumann entropy as $D \to \infty$ are given by:

$$F(G^o_{A|B}) \log D - \mathbb{E} S_n(\rho_A) \xrightarrow[D\to\infty]{} \frac{1}{n-1} \log \left(\int t^n \, \mathrm{d}\mu_{G^o_{A|B}} \right),$$

$$F(G^o_{A|B}) \log D - \mathbb{E} S(\rho_A) \xrightarrow[D\to\infty]{} \int t \, \log t \, \mathrm{d}\mu_{G^o_{A|B}}.$$

where $F(G^o_{A|B}) := \mathrm{maxflow}(G^o_{A|B})$.

Conclusion

Conclusion

From a general random tensor network $|\psi_G\rangle$. Fix $A \subseteq E_\partial$ and $\rho_A = \mathrm{Tr}_B |\psi_G\rangle\langle\psi_G|$ the associated quantum state.

- The moments are given by:

$$\mathbb{E}\,\mathrm{Tr}(\rho_A^n) \sim \sum_{\alpha=(\alpha_x)\in\mathcal{S}_n^{|V|}} D^{-H_G^{(n)}(\alpha)}.$$

- The moment $m_{n,A}^{(D)}$ converges to a graph dependent measure $\mu_{G^o_{A|B}}$ if the obtained ordered graph $G^o_{A|B}$ is series-parallel:

$$m_{n,A}^{(D)} := \frac{1}{D^{F(G^o_{A|B})}}\mathbb{E}\Big[\,\mathrm{Tr}\left(\left(D^{F(G^o_{A|B})}\,\rho_A\right)^n\right)\Big] \xrightarrow[D\to\infty]{} m_{n,G^o_{A|B}}.$$

- The correction terms of the Rényi and Von Neumann entanglement entropy (as $D\to\infty$) are graph dependent (if $G^o_{A|B}$ is series-parallel) given by:

$$\mathbb{E}S_n(\rho_A) \sim F(G^o_{A|B})\log D - \frac{1}{n-1}\log\left(\int t^n\,\mathrm{d}\mu_{G^o_{A|B}}\right),$$

$$\mathbb{E}S(\rho_A) \sim F(G^o_{A|B})\log D - \int t\,\log t\,\mathrm{d}\mu_{G^o_{A|B}}.$$

References