Measurement-induced phase transitions in gaussian fermions and matrix product states

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Outline of the presentation

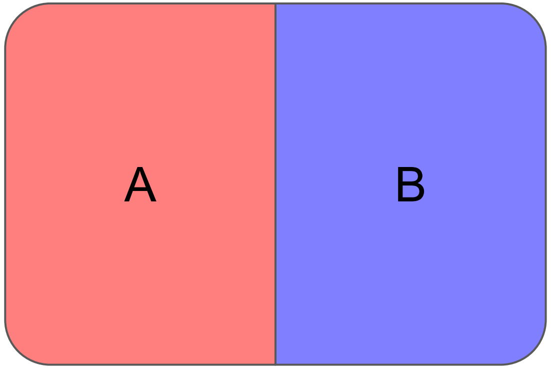
1. Introduction to hybrid quantum circuits
2. Introduction and motivation to measurement-induced phase transitions (MIPTs)
3. The free fermionic case
   a. Introduction
   b. Parity conserving model
   c. Charge conserving model
4. Interacting models with matrix product states (MPS)
   a. Introduction to MPS
   b. The time-dependent variational principle
   c. MIPT as a transition in classical simulatibility
5. Future perspectives and conclusion

What distinguishes **quantum** from **classical** information?

Quantum entanglement

System cannot be described by the state of its components in isolation

How to quantify?



$$\rho_A = \mathrm{Tr}_B(\rho)$$

Rényi entropy

$$S_A^n = \frac{1}{1-n} \log\left[\mathrm{Tr}\left(\rho_A^n\right)\right]$$

Von Neumann entanglement entropy

$$S_A^1 = S_A = -\mathrm{Tr}\left(\rho_A \log \rho_A\right)$$

Is entanglement **extensive**?

$$S_A \propto \quad V_A \quad \log V_A \quad V_{\partial A}$$

Volume law	**Log (critical)** law	**Area** law

Entanglement growth in **quantum circuits**

Why quantum circuits?

- Fundamental in quantum computing
- **Trotterization** of local Hamiltonian evolution

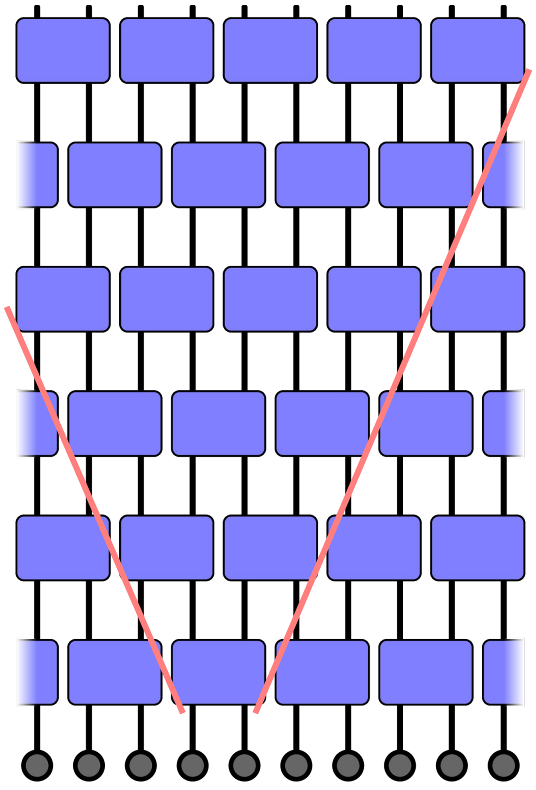
$$\hat{H} = \sum_i \hat{h}_{i,i+1}$$

First order Suzuki–Trotter expansion

$$\begin{aligned}\hat{U} &= e^{-i\delta t \hat{H}} \\ &= e^{-i\delta t \hat{H}_{\text{odd}}} e^{-i\delta t \hat{H}_{\text{even}}} + \mathcal{O}(\delta t^2)\end{aligned}$$

Ballistic growth of entanglement until saturation at **volume**-law

Unitary circuit



$|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle\ldots$

Evolution with random **monitoring**

With probability $p = \gamma \delta t$

Measurement rate

Measure operator $\hat{O} = \sum_k o_k \hat{P}_k$ with the **Born rule**

$$|\psi\rangle \to \frac{\hat{P}_k |\psi\rangle}{\sqrt{p_k}}$$ with probability $p_k = \langle \psi | \hat{P}_k | \psi \rangle$

Projective (strong) measurements

In the continuous time limit $\delta t \to 0$

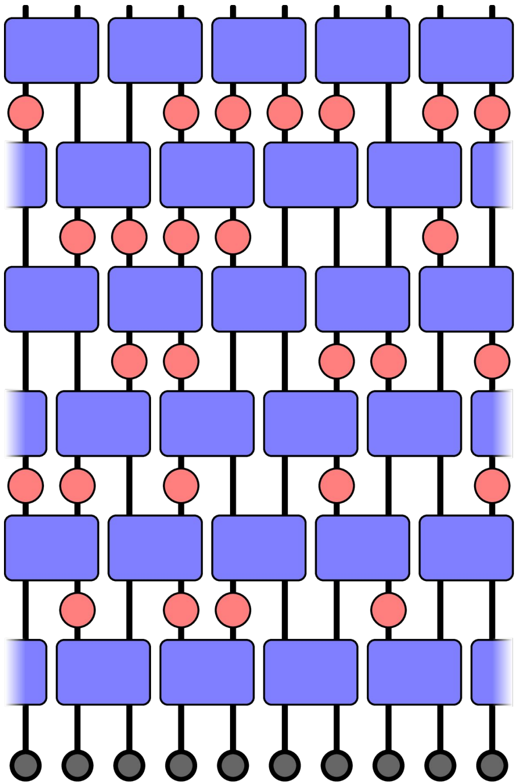
Stochastic Schödinger equation (SSH)

$$d|\psi_t\rangle = -iHdt\,|\psi_t\rangle + \sum_i \left[\sqrt{\gamma}(\hat{O}_i - \langle \hat{O}_i \rangle_t) dW_t^i - \frac{\gamma}{2}(\hat{O}_i - \langle \hat{O}_i \rangle)^2 dt\right] |\psi_t\rangle$$

Continuous (weak) measurements

Average entanglement over **quantum trajectories**

Hybrid circuit



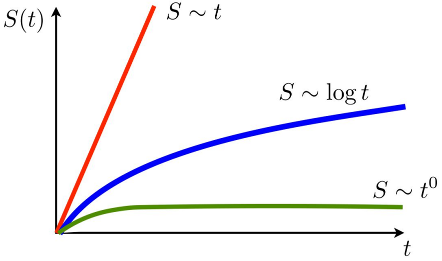
Measurement-induced **entanglement** phase transition

Trajectory-averaged **nonlinear** functions of the state

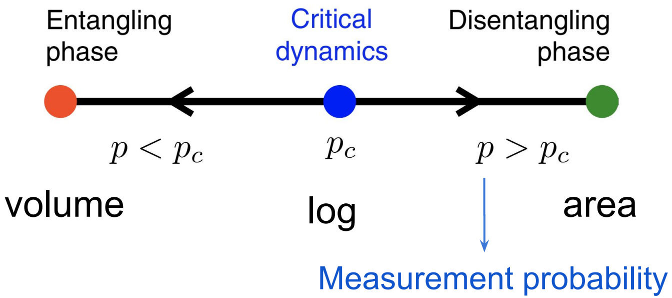
Measurement-induced phase transitions (MIPTs)

$$S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

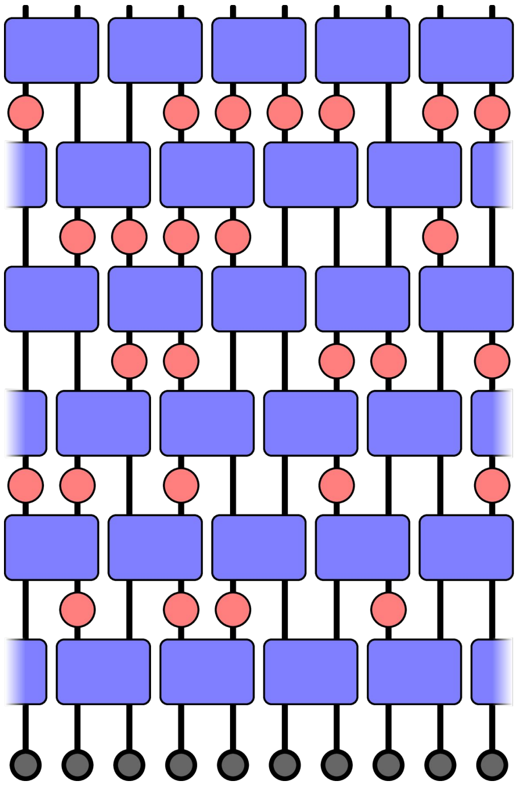
Entanglement growth



Saturated entanglement laws



Hybrid circuit



Measurement-Induced Phase Transitions in the Dynamics of Entanglement, Brian Skinner, Jonathan Ruhman, and Adam Nahum, Phys. Rev. X 9, 031009 (2019)

Measurement-driven entanglement transition in hybrid quantum circuits, Yaodong Li, Xiao Chen, and Matthew P. A. Fisher, Phys. Rev. B 100, 134306 (2019)

Measurement-induced **purification** phase transition

What if the initial state is **mixed**? Ex: $\rho_0 = \frac{1}{2L}\mathbb{1}$

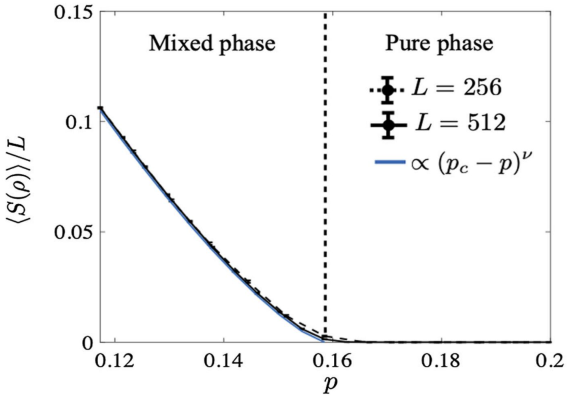
System size

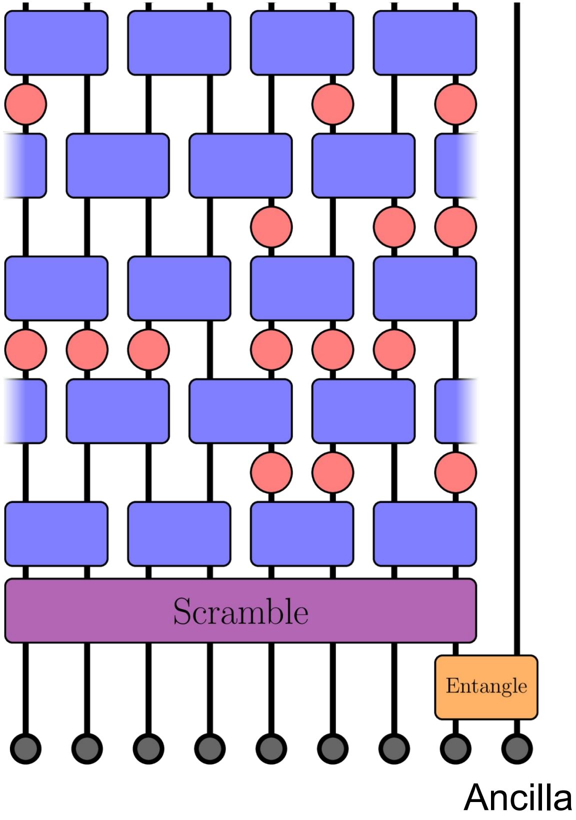
Purification timescale

$p < p_c$ ⇒ τ_P superlinear in L

$p > p_c$ ⇒ τ_P sublinear in L

Residual Von Neumann entropy at $t = 4L$





Probe residual Ancilla entropy

Dynamical Purification Phase Transition Induced by Quantum Measurements,
Michael J. Gullans and David A. Huse, Phys. Rev. X 10, 041020 (2020)

Why are MIPTs **interesting**?

- Monitoring in quantum trajectories can describe the evolution of **open quantum systems** by unravelling the Linbladian
- Connection to **quantum error correction** and quantum channel capacity
- A replica trick approach to random hybrid circuits can map the MIPT to a ground state problem in an **effective spin model** (universality of dynamical phase transitions)
- The MIPT can generally be viewed as a **classical simulatability** transition

→ Matrix Product States (MPS)

The **experimental** challenge

Finding identical quantum trajectories for tomography is exponentially unlikely in circuit depth → **Post-selection problem**

Small number of qubits

Measurement-induced quantum phases realized in a trapped-ion quantum computer, Crystal Noel et al. Nature Physics volume 18, pages 760–764 (2022)
Experimental Realization of a Measurement-Induced Entanglement Phase Transition on a Superconducting Quantum Processor, Jin Ming Koh et al. arXiv:2203.04338 (2022)

Free fermionic models

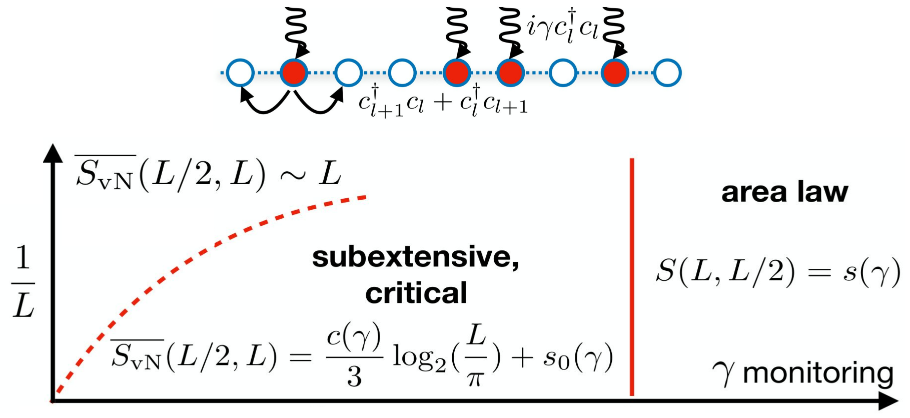
Free fermionic models

No interactions means that the states remain **Gaussian on each trajectory** and everything can be computed from the correlation matrix (Wick's theorem)

$$C_{a,b} = \mathrm{tr}(\rho \hat{c}_a^\dagger \hat{c}_b)$$

Polynomial scaling with L

Classically simulatable



With

$$\gamma > 0$$

Transition from extensive **critical** log phase to area law

or

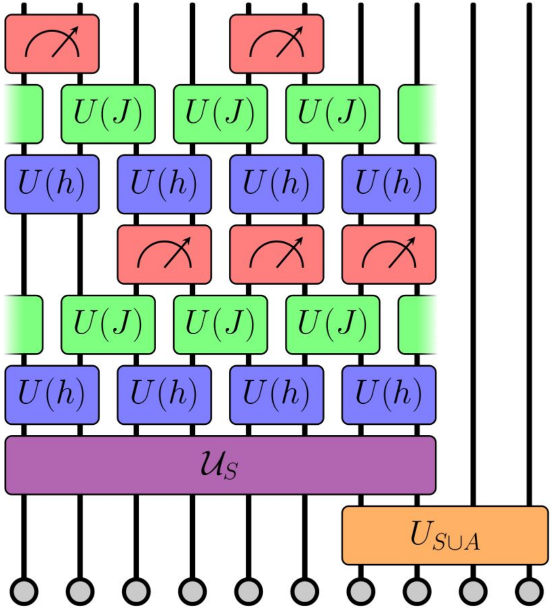
No transition (area law)

The MIPT is **symmetry** dependent

Entanglement Transition in a Monitored Free-Fermion Chain: From Extended Criticality to Area Law, O. Alberton, M. Buchhold, and S. Diehl, Phys. Rev. Lett. 126, 170602 (2021)

Symmetry preserving free fermionic models

$\mathbb{Z}_2$ symmetry (**parity** conserving)



Trotterized transverse field Ising model

$$\hat{U}_{a,b}(\alpha) = \exp(-\alpha\hat{\gamma}_a\hat{\gamma}_b)$$

Majorana fermions

$$\{\hat{\gamma}_a, \hat{\gamma}_b\} = 2\delta_{ab}$$

Dirac fermions

$$\{\hat{c}_a, \hat{c}_b^\dagger\} = \delta_{ab}$$

$$\hat{\gamma}_{2a} = -i\left(\hat{c}_a^\dagger - \hat{c}_a\right) \ , \ \hat{\gamma}_{2a-1} = \hat{c}_a + \hat{c}_a^\dagger$$

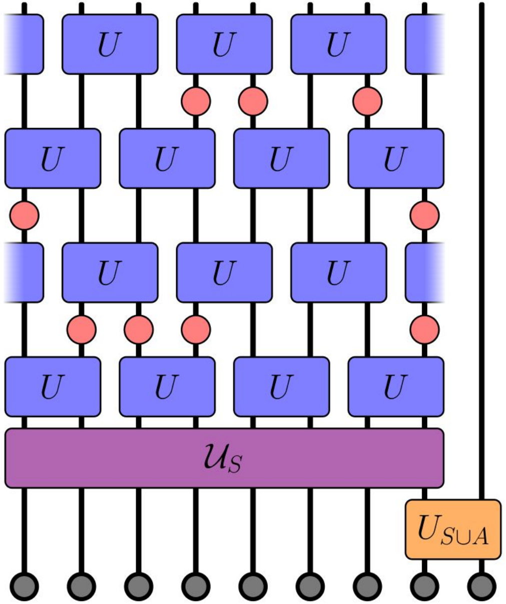
Measure the **density** operator

$$\hat{n}_a = \hat{c}_a^\dagger \hat{c}_a$$

Initial state

$$|\psi_0\rangle = \hat{\mathcal{U}}_S \hat{U}_{S\cup A} |\psi_{S,0}\rangle \otimes |\psi_{A,0}\rangle$$

$\mathrm{U}(1)$ symmetry (**charge** conserving)



Random U(1) unitary gates

$$\hat{U}_{a,b} = e^{-2i\beta(\hat{c}_a^\dagger\hat{c}_b + \mathrm{h.c.})}$$

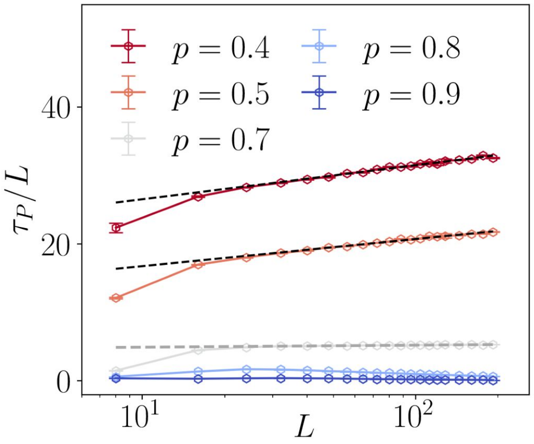
$$\beta \sim U[0,\pi]$$

Measurement-induced **purification** transitions

$\mathbb{Z}_2$ symmetry (**parity** conserving)

$p < p_c$ ⇒ $\tau_P = \mathcal{O}(L \ln L)$

$p > p_c$ ⇒ $\tau_P = \mathcal{O}(\ln L)$



Purity revealed by

$$\langle S_2[\rho_S(t)] \rangle$$

$$S_2[\rho] = -\log[\mathrm{Tr}(\rho^2)]$$

Purification **timescale**

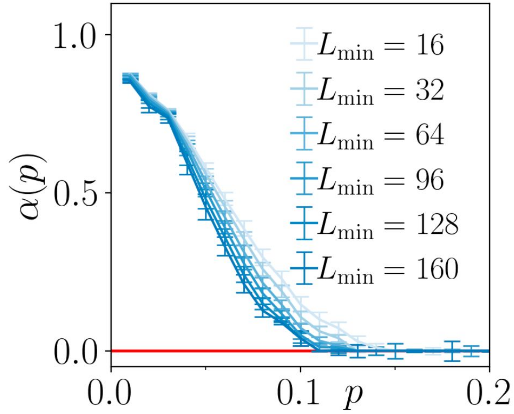
$$t_P = \min_t \{t : S_2(t) \simeq 0\}$$

$$\tau_P = \mathrm{median}(t_P)$$

$\mathrm{U}(1)$ symmetry (**charge** conserving)

$p < p_c$ ⇒ $\tau_P = \mathcal{O}(L^{\alpha(p)})$

$p > p_c$ ⇒ $\tau_P = \mathcal{O}(\ln L)$



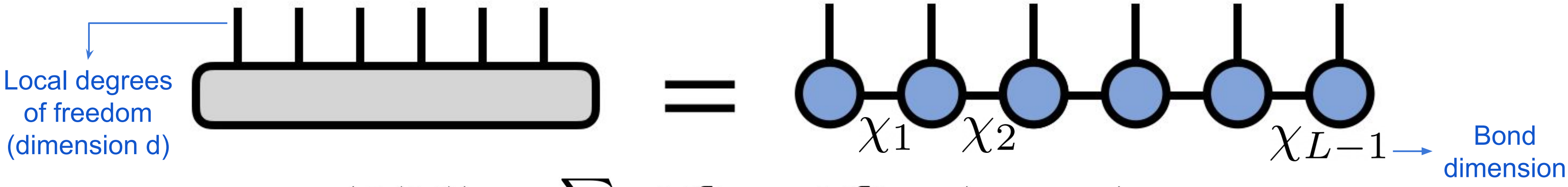
Nonlinear Sigma Models for Monitored Dynamics of Free Fermions, Michele Fava, Lorenzo Piroli, Tobias Swann, Denis Bernard, and Adam Nahum, Phys. Rev. X 13, 041045 (2023) → Effective non-linear sigma models (NLSM) ← **Theory of Free Fermions under Random Projective Measurements,** Igor Poboiko, Paul Pöpperl, Igor V. Gornyi, and Alexander D. Mirlin, Phys. Rev. X 13, 041046 (2023)

→ Area law for exponentially large systems

Coming soon: NLSM derivation in clustered Brownian SYK models

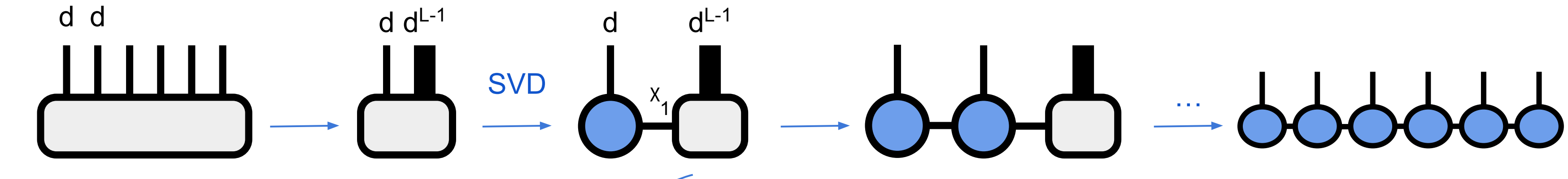
The interacting case

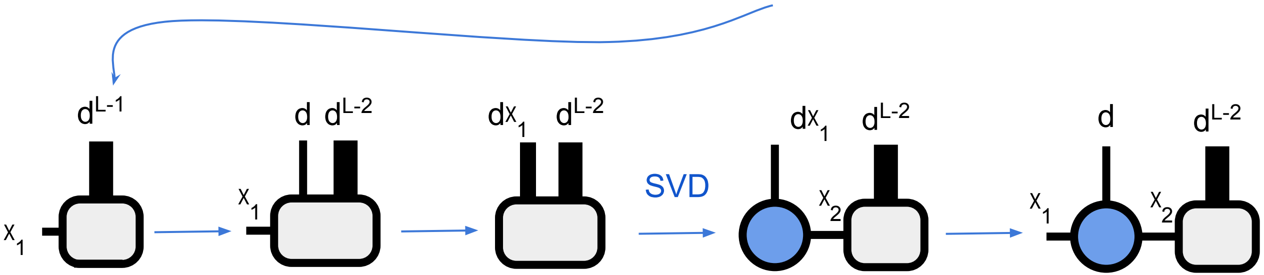
Quick introduction to **matrix product states**



$$|\Psi(M)\rangle = \sum_{\sigma_1,\ldots,\sigma_L} M^{\sigma_1}_{1;\chi_1} \cdots M^{\sigma_L}_{L;\chi_{L-1}} |\sigma_1 \cdots \sigma_L\rangle$$

How to **construct** the MPS?





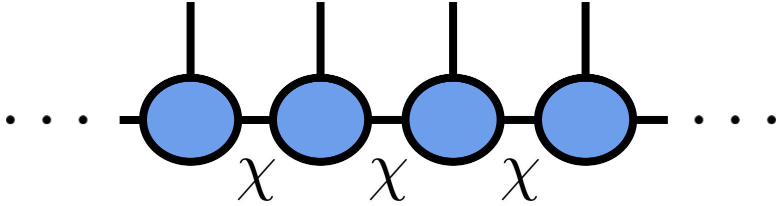
Exponential growth of the bond dimension

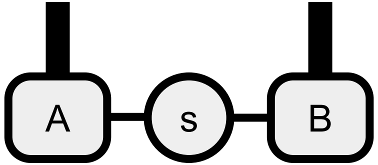
$$\chi_1 = d$$
$$\chi_2 = d^2$$
$$\vdots$$
$$\chi_n = d^n$$

Can we **compress** the MPS?

Fix maximal **bond dimension** $\chi_i = \chi$

(truncate smaller Schmidt values after SVD)





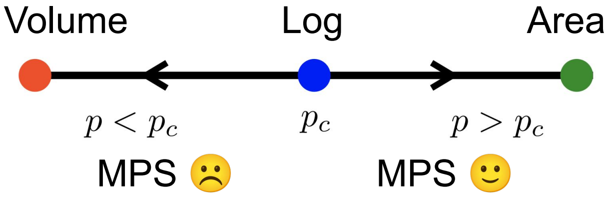
$$|\psi\rangle = \sum_i s_i \left|\psi_i^A\right\rangle \otimes \left|\psi_i^B\right\rangle \Rightarrow S_A = -\sum_i s_i^2 \log s_i^2$$

Entanglement **upper bound** at fixed bond dimension χ $\quad S_A \leq \log \chi$

There is always a **finite** χ such that

$$\left\| |\psi_{\mathrm{Area}}\rangle - |\psi_{\mathrm{MPS}}\rangle \right\|^2 < \varepsilon \quad \forall \quad \varepsilon > 0$$

MIPT as a **simulatibily** transition



Efficient numerical simulations using matrix-product states, Frank Pollmann (2016)

The Time-Dependent Variational Principle (**TDVP**)

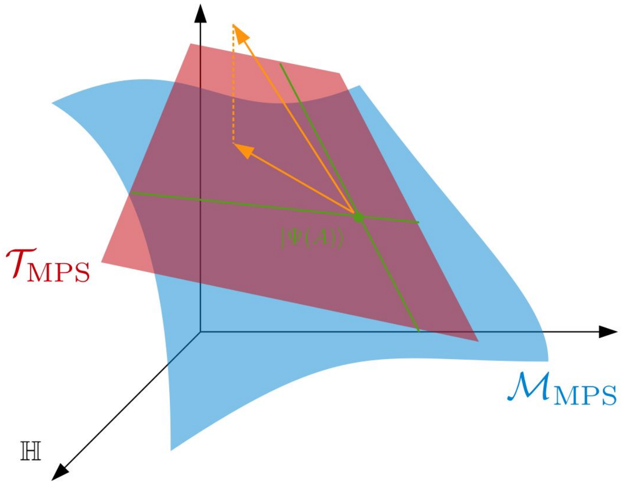
Time evolution with **projection** to MPS manifold

$$i\partial_t|\Psi(M)\rangle = P_{\mathcal{T}_M}\hat{H}|\Psi(M)\rangle$$

Tangent space of fixed χ

Effective nonlinear **symplectic** evolution
(unitary within the manifold)

Have the same **conservation** laws of the exact dynamics



Error rate $$E(\chi) = \|\hat{H}\,|\psi\rangle - P_{\mathcal{T}_{M_\chi}}\hat{H}\,|\psi\rangle\,\|^2$$

Volume law ⟶ $E(\chi) \sim \dfrac{1}{\log(\chi)}$

Area law ⟶ $E(\chi) \sim e^{-\chi}$

Unifying time evolution and optimization with matrix product states, Jutho Haegeman, Christian Lubich, Ivan Oseledets, Bart Vandereycken, Frank Verstraete, arXiv:1408.5056 (2015)

$\mathrm{U}(1)$ symmetric **interacting** model

Example: **XXX** spin chain $$\hat{H}_{\mathrm{XXX}} = \sum_{i=1}^{L} \left(\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \hat{S}_i^z \hat{S}_{i+1}^z \right)$$

Measure local magnetization $\hat{S}_i^z$ with a measurement rate γ

Continuous measurements - **stochastic Schödinger equation**

$$d\,|\psi_t\rangle = -iH\,dt\,|\psi_t\rangle + \sum_{i=1}^{L} \left[\sqrt{\gamma}(\hat{S}_i^z - \langle \hat{S}_i^z \rangle_t)dW_t^i - \frac{\gamma}{2}(\hat{S}_i^z - \langle \hat{S}_i^z \rangle_t)^2 dt \right] |\psi_t\rangle$$

$$|\psi_{t+\delta t}\rangle \approx C e^{\sum_{j=1}^{L} \left[\delta W_t^j + 2\langle \hat{S}_j^z \rangle_t \gamma \delta t\right] \hat{S}_j^z} e^{-\mathbf{i}\hat{H}\delta t} \, |\psi_t\rangle \qquad \delta W_t^j \sim \mathcal{N}(0, \gamma\delta t)$$

Single site operators, no truncation

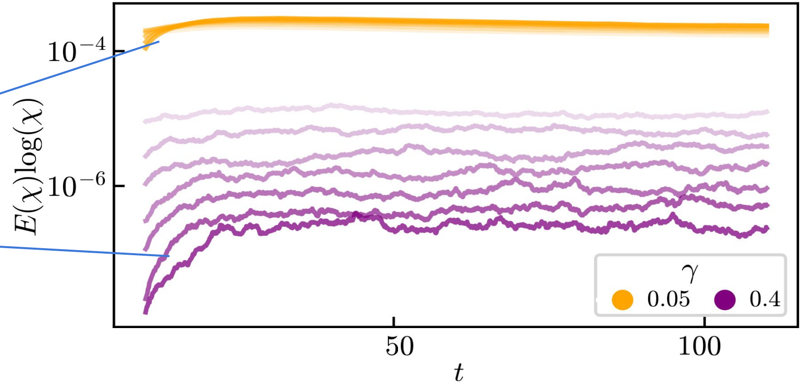
TDVP algorithm

Probing the **entanglement** transition with the TDVP error

Large time behaviour of the error rate

Constant value in the volume law

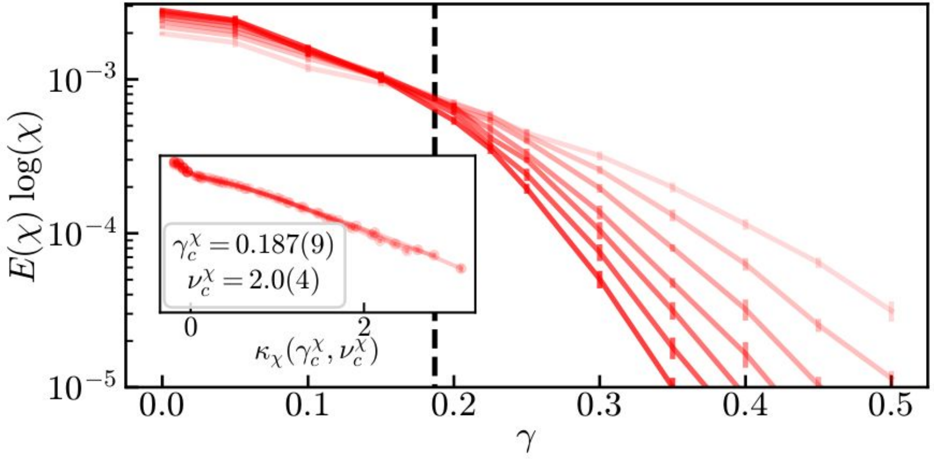
Exponentially decreasing in the area law



Time-average after saturation

Critical parameters from finite-size scaling analysis

$$\kappa_A(\gamma_{\mathcal{O}}, \nu_{\mathcal{O}}) = \begin{cases} (\gamma - \gamma_{\mathcal{O}}) & \gamma < \gamma_{\mathcal{O}} \\ (\gamma - \gamma_{\mathcal{O}}) A^{1/\nu_{\mathcal{O}}} & \gamma > \gamma_{\mathcal{O}} \end{cases}$$



We detected the MIPT by employing the "wrong"order of limit!

1. $t \to \infty$
2. $\chi \to \infty$

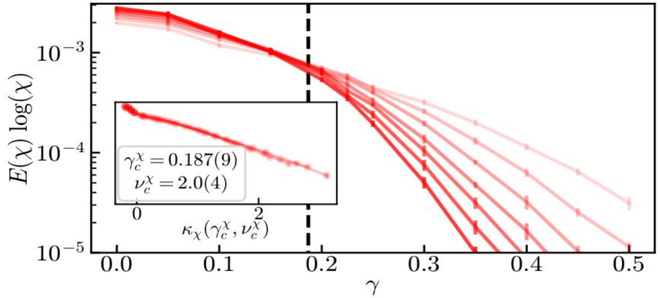
Comparison with **exact diagonalization**

Trotter decomposition of **unitary** evolution

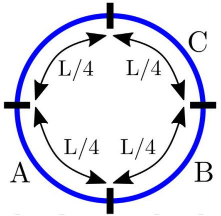
$$\hat{U}_{\mathrm{XXX}} \approx \prod_i \hat{u}^{\mathrm{XXX}}_{2i,2i+1} \prod_i \hat{u}^{\mathrm{XXX}}_{2i-1,2i} \qquad \hat{u}^{\mathrm{XXX}}_{i,i+1} = \exp\left[-i\delta t\left(\hat{S}^x_i \hat{S}^x_{i+1} + \hat{S}^y_i \hat{S}^y_{i+1} + \hat{S}^z_i \hat{S}^z_{i+1}\right)\right]$$

Projective Z measurements for the **monitoring** $P_\pm = \left(\frac{\mathbb{1}}{2} \pm \hat{S}^z_i\right)$

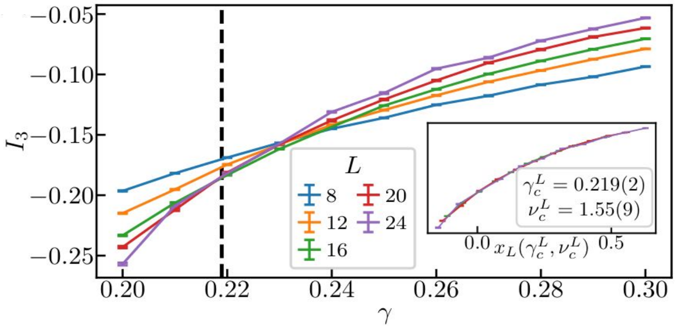
$|\psi\rangle \to P_\pm |\psi\rangle$ with probability $p_\pm = \langle\psi| P_\pm |\psi\rangle$



Tripartite mutual information



$$\mathcal{I}_{3,n}(A,B,C) \equiv S_n(A) + S_n(B) + S_n(C) - S_n(A\cup B) - S_n(A\cup C) - S_n(B\cup C) + S_n(A\cup B\cup C)$$



$x_A(\gamma_\mathcal{O}, \nu_\mathcal{O}) = (\gamma - \gamma_\mathcal{O})A^{1/\nu_\mathcal{O}}$

Expect γ^L_c to **drift left** in the thermodynamic limit

Predict $\gamma_c \simeq 0.19$

Measurement-induced transitions beyond Gaussianity: A single particle description,
Luca Lumia, Emanuele Tirrito, Rosario Fazio, and Mario Collura, PRR 023176 (2024)

Future directions

Can MPS help us solve the **post-selection** problem?

How to **detect** the post-measurement state with the least amount of copies?

Shadow tomography $\rho \xrightarrow{\text{Random unitary}} U\rho U^\dagger \xrightarrow{\text{measure Z}} \underbrace{\rho_r^S}_{\text{Shadow}} = 3U_r^{-1}|z_r\rangle\langle z_r|U_r - \mathbb{1} \Rightarrow \mathbb{E}_r[\rho_r^S] = \rho$

$\mathcal{O}(\log L)$ measurements for local observables

$\mathcal{O}(2^{V_A})$ measurements for S_A

The randomized measurement toolbox, Andreas Elben et al. Nature Review Physics 5, 9-24 (2023)

Idea: Quantum-Classical **correlations** as a non-linear observable

$$S_r^{SC} = -\mathrm{Tr}[\rho_r^S \log \underbrace{\rho_{m_r}^C}_{\text{Classical}}] \Rightarrow \mathbb{E}_m[S_m] \leq \mathbb{E}_r[S_r^{SC}]$$

Probing post-measurement entanglement without post-selection, Samuel J. Garratt and Ehud Altman, arXiv:2305.20092 (2023)

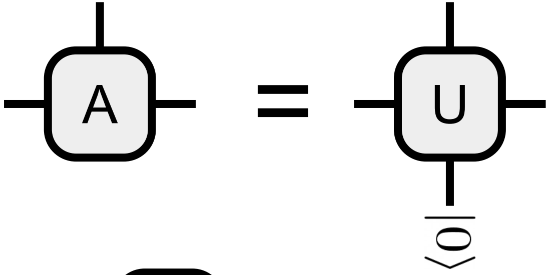
Can we follow a similar protocol for the **error**?

$$E_m(T,\chi) = 1 - \langle\psi_m|\psi_m(\chi)\rangle \xrightarrow{\text{shadow}} E_m(T,\chi) = 1 - \frac{\underbrace{m_T}_{\text{Shadow}} \cdot \underbrace{m_T(\chi)}_{\text{Classical}}}{||m_T||^2}$$

Advantage: Do not have to reconstruct the density matrix (exponentially large)

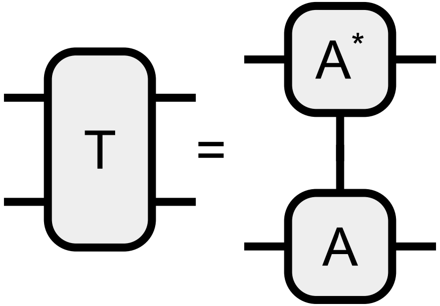
Describing these **ensembles** of MPSs

We expect ergodic unitary dynamics to produce **random MPSs**

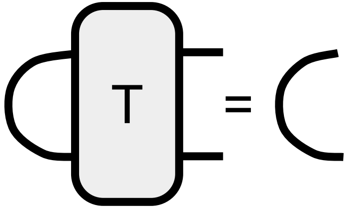


$U \longrightarrow$ Haar Random unitary

Transfer matrix



By construction, the highest eigenvalue is one



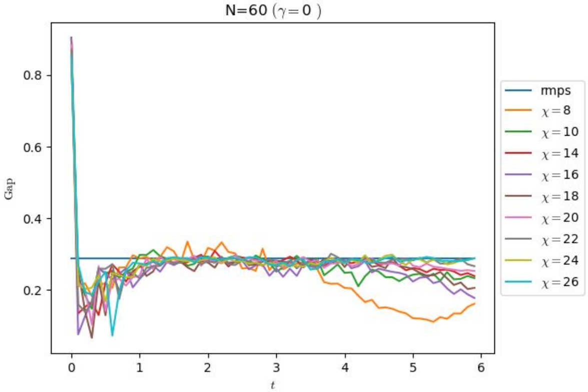
We can look at the **gap**

Random quantum channel

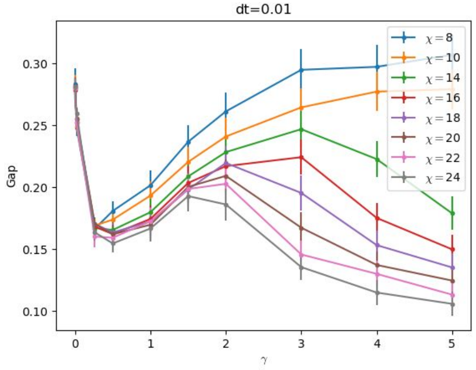
$$\mathrm{gap} = 1 - \frac{1}{\sqrt{d}}$$

Generating random quantum channels, Ryszard Kukulski, Ion Nechita, Łukasz Pawela, Zbigniew Puchała, Karol Życzkowski, J. Math. Phys. 62, 062201 (2021)

TDVP interacting spin-chain



TDVP interacting spin-chain + continuous monitoring



Conclusions

- Purification transition from **superlinear to sublinear** purification timescales in the free fermionic parity conserving model
- Purification transition always with **sublinear** purification in the charge conserving free fermionic model, subject to finite-size effects
- Successfully probed MIPTs from the **error rate** of the **TDVP** method in interacting charge conserving systems
- Checked MPS results with exact diagonalization simulations

Future works

- Solve the post-selection problem with a quantum-classical error rate?
- Can we describe the ensemble of MPSs for different rates of measurements?

Purification timescales in monitored fermions, Hugo Lóio, Andrea De Luca, Jacopo De Nardis, and Xhek Turkeshi, Phys. Rev. B 108, L020306 (2023)
Measurement-induced phase transitions by matrix product states scaling, Guillaume Cecile, Hugo Lóio, Jacopo De Nardis, arXiv:2402.13160 (2024)

Special **thanks** to

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Andrea De Luca

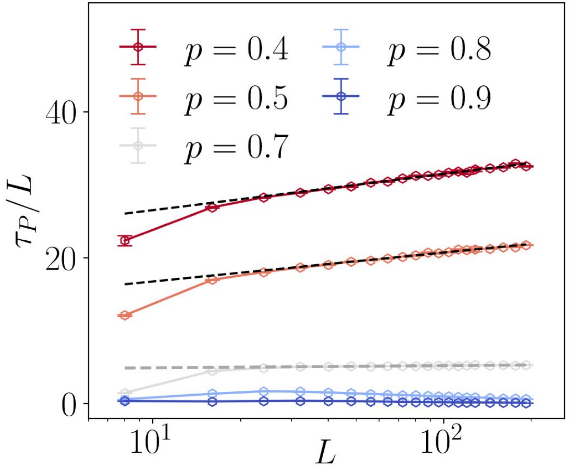
Xhek Turkeshi

Guillaume Cecile

Backup

MIPT in the **Majorana** circuit

$$|\psi_0\rangle = \hat{\mathcal{U}}_S \hat{U}_{S\cup A}\,|\psi_{S,0}\rangle \otimes |\psi_{A,0}\rangle$$



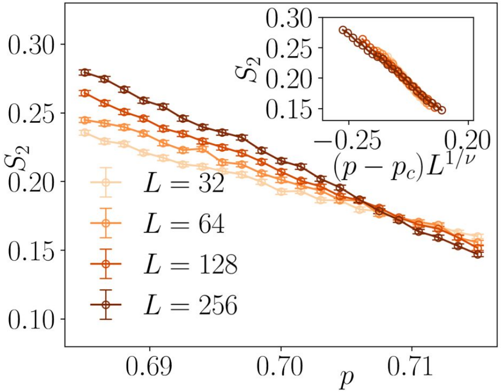
Purity revealed by $\langle S_2[\rho_S(t)]\rangle$

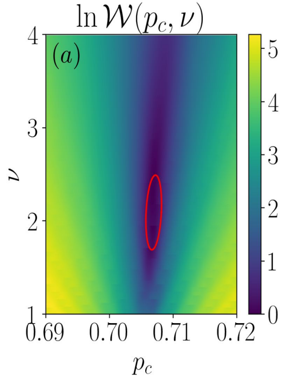
$$\tau_P = \mathrm{median}(t_P) \qquad t_P = \min_t\{t : S_2(t) \simeq 0\}$$

$$p < p_c \;\Rightarrow\; \tau_P = \mathcal{O}(L\ln L)$$

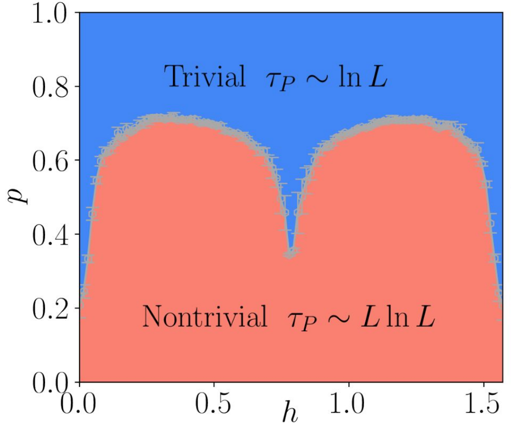
$$p > p_c \;\Rightarrow\; \tau_P = \mathcal{O}(\ln L)$$

$t \;=\; L \quad \rho_{0,S} = \mathbb{1}/2^L$ (no Ancilla)





Phase diagram



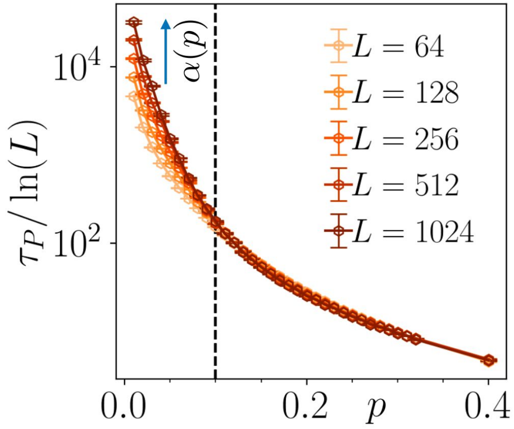
$$h = 0.3 \;\Rightarrow\; p_c = 0.707(3)\;,\;\; \nu = 2.1(4)$$

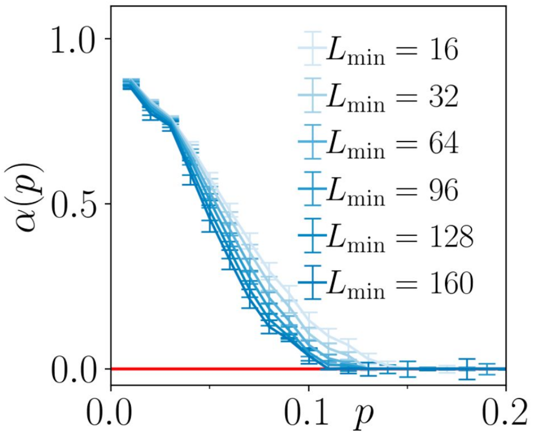
MIPT in the **Dirac** circuit

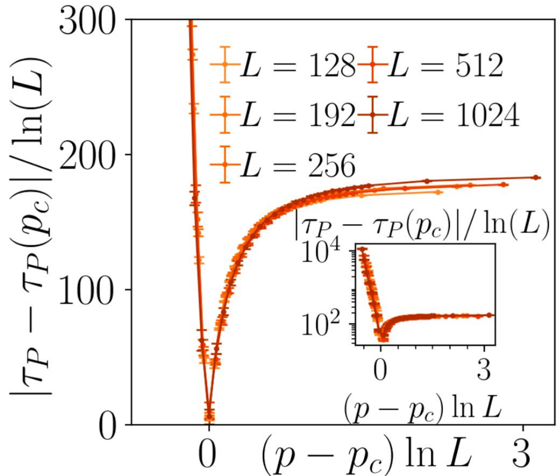
Purification **timescale** is always sublinear

$p < p_c \Rightarrow \tau_P \propto L^{\alpha(p)}$

$p > p_c \Rightarrow \tau_P \propto \ln L$







Data collapse typical of a **BKT** (Berezinskii-Kosterlitz-Thouless) transition

Charge-sharpening transition in the XXX chain

Local charge **variance**

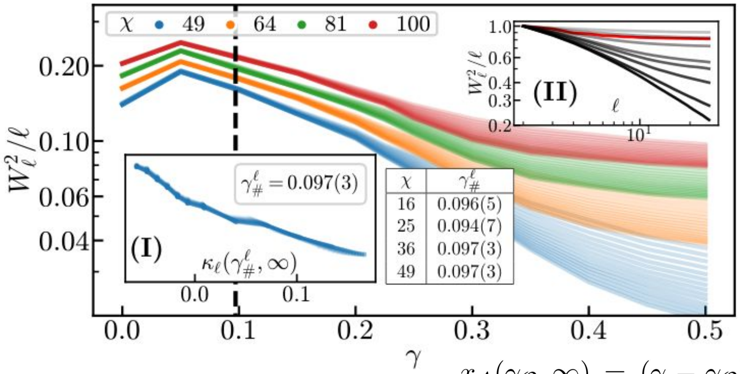
$$Q_\ell = \sum_{j\in\ell} S_j^z$$

$$W_\ell^2 = \langle Q_\ell^2\rangle - \langle Q_\ell\rangle^2 = \sum_{i,j\in\ell} \langle S_i^z S_j^z\rangle^c$$

$\gamma < \gamma_\#$ ⟹ $W_\ell^2 \sim \ell$

$\gamma > \gamma_\#$ ⟹ W_ℓ^2 sublinear

$\gamma_\# < \gamma_c$



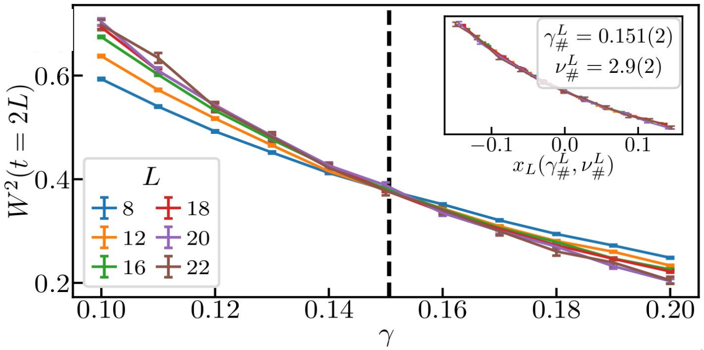
$x_A(\gamma_\mathcal{O}, \infty) = (\gamma - \gamma_\mathcal{O}) \log(A)$

Start with a superposition of all charge sectors

$$|\Psi(0)\rangle = \bigotimes_{i=1}^{L} (|\uparrow\rangle + |\downarrow\rangle)$$

Total charge **variance**

$$W^2(t) = \langle Q^2(t)\rangle - \langle Q(t)\rangle^2$$



$x_A(\gamma_\mathcal{O}, \nu_\mathcal{O}) = (\gamma - \gamma_\mathcal{O}) A^{1/\nu_\mathcal{O}}$

Superlinear
to
sublinear
charge-sharpening timescale transition