Rapid mixing of commuting spin lattice systems

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Thermalization of quantum spin systems: From 'weak' inequalities to strong results

- **Davies thermalization** is a Markovian model for thermalization of spin lattice systems where every lattice point individually couples weakly to the heat bath at temperature β^{-1}.
- This dynamic drives the system to its thermal state, also called the Gibbs state: $\rho_t \xrightarrow{t\to\infty} \sigma_\beta$.
- Questions:
  1. How fast does this happen? $\to t_{\text{mix}}(\epsilon) := \inf\{t \geq 0 | \|\rho_t - \sigma\|_1 \leq \epsilon\}$
     And how does this mixing time scale in system size N?
  2. What is the relation to the locality structure of the system Hamiltonian/ its Gibbs state? $\leftrightarrow$ Notions of clustering in the Gibbs state
- Consider systems on D-**dim hypercubic lattices** $\Lambda \subset \mathbb{Z}^D$ with **geometrically-local**, **commuting**, uniformly bounded Hamiltonians:

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi_X \tag{1}$$

$\Phi_X = 0$ for diam $X > r$

$[\Phi_X, \Phi_Y] = 0 \quad \forall X, Y \subset \Lambda$ → e.g. Toric code,...

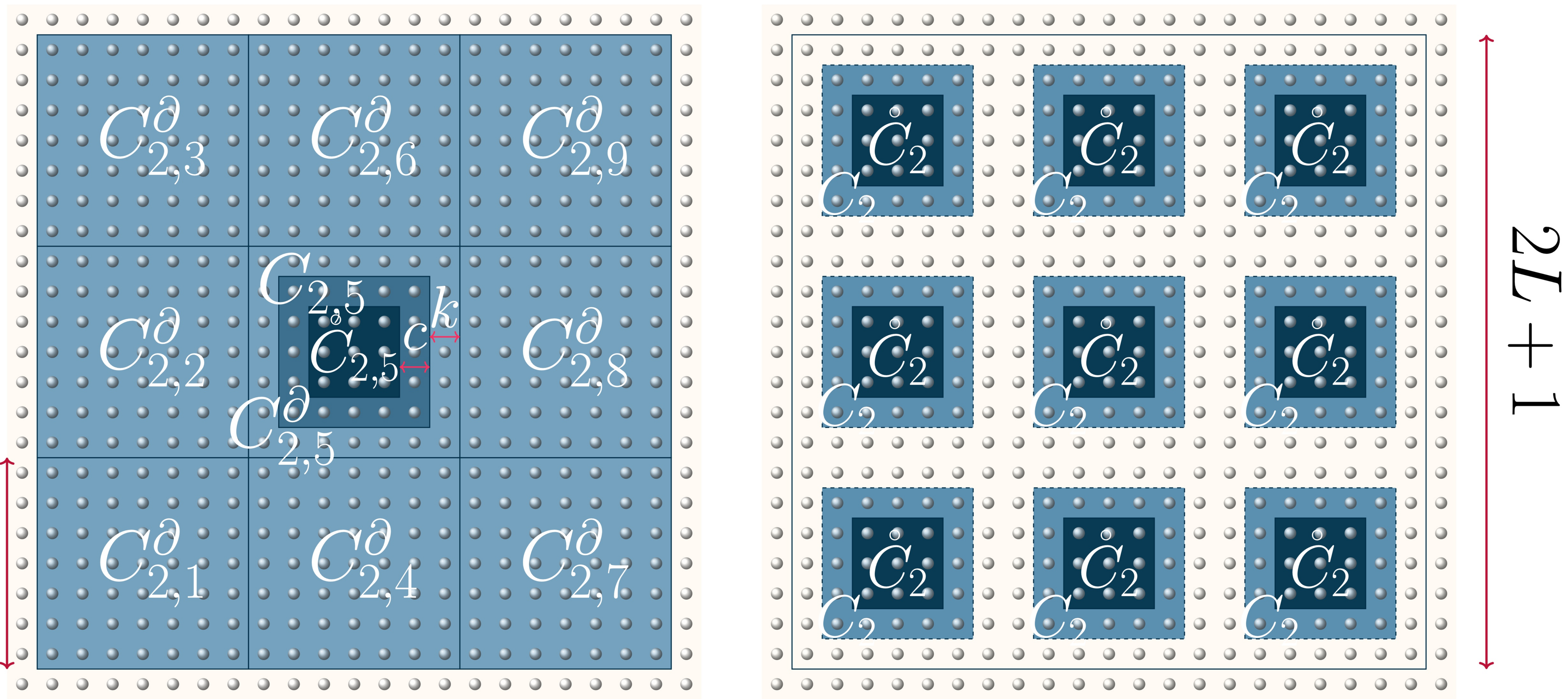


Figure 1. The first layer of a suitable covering of the D-dimensional sublattice $[-L, L]^D$ into almost $D - a$-dim. regions $\{C_{a,i}\}_{a,i}$ size ℓ and overlapp c. It is a generalization of a suitable 2D covering of [2].

Main Results

Let $([-L, L]^D, H_\Lambda, \beta)$ be a **commuting** and **geometrically-local** quantum spin systems on the D-dimensional lattice. Then strong local indistinguishability of the Gibbs state implies that the associated Davies dynamics

1. *Theorem 1:* [Rapid thermalization from sLI & Gap] are rapidly mixing, i.e. $\|e^{t\mathcal{L}^D_\Lambda}(\rho) - \sigma\|_1 \leq \epsilon$ for
$$t \geq t_{\text{mix}}(\epsilon) = \mathcal{O}(\log(N)^D, \log\epsilon^{-1}) \tag{2}$$
when the local gap of the Davies generators $\lambda(\mathcal{L}^D_X)$, for $X \subset \Lambda_L$ scales inverse polynomially with $|X|$.
2. *Theorem 2:* [Quasi-rapid Wasserstein mixing from sLI] satisfy quasi-rapid Wasserstein mixing, i.e. $W_1(e^{t\mathcal{L}^D_\Lambda}(\rho), \sigma) \leq \epsilon N$ for
$$t \geq t_{W_1}(\epsilon) = \mathcal{O}(\exp(\text{poly}\log\log(N)), \log\epsilon^{-1}), \tag{3}$$
when assuming any strictly positive local gap $\lambda(\mathcal{L}^D_X) > 0$. We have set $N := |\Lambda|$

an extensive distance that quantifies local differences

I. Strong Local indistinguishability (sLI)

Global thermalization results rely on **lifting local rapid thermalization** to the whole lattice [3-5]. Such arguments need a notion of **spacial clustering**, i.e. stability of the Gibbs state in some region of interest A to changes of the Hamiltonian 'far away' C. To quantify this we find the notion of strong local indistinguishability very useful.

- In 1D systems of form (1) it implies rapid mixing at any temperature $\beta > 0$ [1].

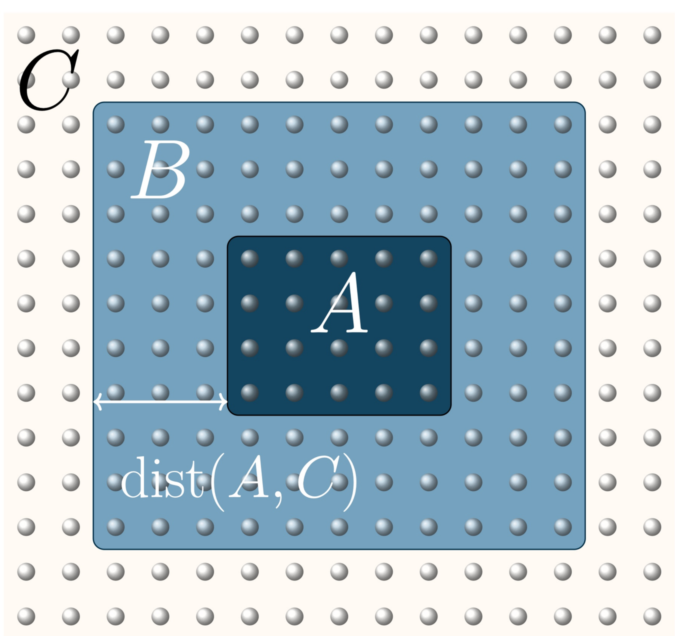


Figure 2. A region A shielded from a large region C by a ring B, as in the definition of sLI (4).

Definition:

$\sigma_A^X \propto \text{tr}_{A^c}[e^{-\beta H_X}]$

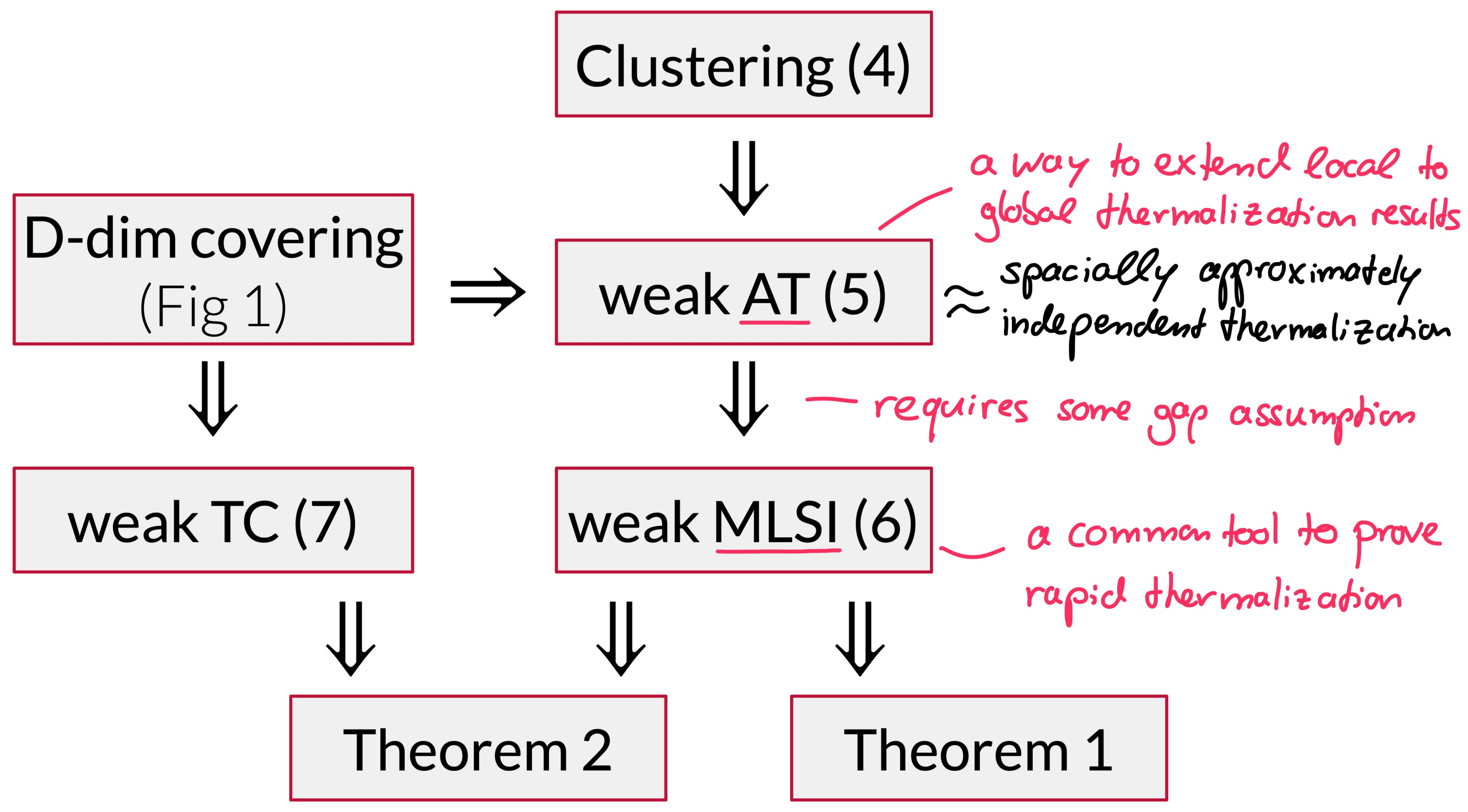
exponentially decaying!

$$D_{\max}(\sigma_A^{ABC} \| \sigma_A^{AB}) \leq K \max\{|A|, |C|\}^\nu \exp\left(-\frac{\text{dist}(A, C)}{\xi}\right) \tag{4}$$

$D_{\max}(\rho\|\sigma) := \log\|\sigma^{-1/2}\rho\sigma^{-1/2}\|$

References: [1] Kochanowski, Alhambra, Capel, Rouzé; 2024, [2] Brandão, Kastoryano; 2019, [3] Bardet et al. 2021, [4] Capel et al. 2021, [5] Zegarlinski 1994, [6] Onorati et al. 2023, [7] Chen et al. 2023 & Ding et al. 2024

II. Proof idea and tools



- Derive a weak approximate tensorization statement
$$D(\rho\|\sigma^{\Lambda_L}) \leq \sum_{a,i} D(\rho\|E_{C_{a,i}}(\rho)) + \epsilon \tag{5}$$
- Derive a weak modified logarithmic Sobolev inequality
$$D(\rho\|\sigma^{\Lambda_L}) \leq \mathcal{O}\left(\left(\log N\epsilon^{-1}\right)^D\right) \text{EP}_\Lambda(\rho) + \epsilon \tag{6}$$
- Extend a weak transport cost inequality to $D-$dimensions from [6]
$$W_1(\rho, \sigma^{\Lambda_L}) \leq \mathcal{O}\left(\log\left(N\delta^{-1/2}\right)^D\right)\sqrt{ND(\rho\|\sigma)} + \mathcal{O}(\delta) \tag{7}$$

Applications and Conclusion

- Efficient preparation of Gibbs states of geom-local commuting Hamiltonians that satisfy sLI and a polynomial gap
- Efficient estimation of local observables through W_1-decay under much weaker assumption of gap
- Efficient estimation of quantum partition functions of such systems via rapid mixing
- No go results of such systems as quantum memory devices

Outlook:

- Rapid mixing of non GNS-symmetric and non-commuting Lindblad generators of [7] → efficient preparation and sampling of very general class of Gibbs states.
- Improved transport cost inequality (7) would give almost constant W_1-mixing time.