The Quantum Wasserstein Distance of Order 1

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Annales Henri Poincaré 23, 3391 (2022)

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PRX Quantum 4, 010309 (2023)

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Annales Henri Poincaré 24, 4237 (2023)

Quantum spin systems

- Finite set of spins *Λ* endowed with distance
- Associate to each spin *x*∈*Λ* the local Hilbert space $\mathcal{H}_x = \mathbb{C}^d$
- Hamiltonian with finite-range interactions
- Gibbs state $$\omega = \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$$
- If correlations decay sufficiently fast, *ω* satisfies TCI

$$\frac{1}{n}\,\|\rho - \omega\|_{W_1} \leq \sqrt{\frac{O(1)}{n}\,S(\rho\|\omega)}$$

- Holds at high enough temperature for any finite-range commuting Hamiltonian [see also Onorati, Rouzé, França, Watson, arXiv:2301.12946]

Equivalence of ensembles

- Canonical ensemble: Gibbs states
- Microcanonical ensemble: uniform convex combination of all states in energy shell
- Assume that Gibbs state satisfies TCI

$$\frac{1}{n}\|\rho-\omega\|_{W_1} \leq \sqrt{\frac{c}{n} S(\rho\|\omega)}$$

- Then, any ρ with same average energy as ω and approximately same entropy as ω is close to ω

$$\mathrm{Tr}\,[\rho\, H] = \mathrm{Tr}\,[\omega\, H] \quad \Longrightarrow \quad \frac{1}{n}\|\rho-\omega\|_{W_1} \leq \sqrt{\frac{c}{n}\left(S(\omega)-S(\rho)\right)}$$

- Ok if fraction of states in shell is

$$e^{-o(n)}$$

Quantum spin systems on $\mathbb{Z}^D$

- Associate to each $x \in \mathbb{Z}^d$ local Hilbert space $\mathcal{H}_x = \mathbb{C}^d$
- Associate to each $\Lambda \subset\subset \mathbb{Z}^D$ the Hilbert space

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$$

- Algebra of operators acting on *Λ*: $\mathfrak{U}_\Lambda$
- Local algebra

$$\mathfrak{U}_{\mathbb{Z}^D} = \overline{\bigcup_{\Lambda\subset\subset\mathbb{Z}^D} \mathfrak{U}_\Lambda}$$

- Quantum state: Positive unital linear functional on $\mathfrak{U}_{\mathbb{Z}^D}$
- We consider translation-invariant states
- Marginal states $\rho_\Lambda \in \mathcal{S}_\Lambda : \mathrm{Tr}_{\mathcal{H}_\Lambda}[\rho_\Lambda A] = \rho(A) \qquad \forall\, A \in \mathfrak{U}_\Lambda$

Interactions

- Interaction: collection of observables $\{h_\Lambda \in \mathcal{O}_\Lambda\}_{\Lambda\subset\subset\mathbb{Z}^D}$
- Hamiltonian of region *Λ*: $H_\Lambda^h = \sum_{X\subseteq\Lambda} h_X$
- We consider translation-invariant interactions with finite local norm

$$\|h\|_r = \sum_{0\in\Lambda\subset\subset\mathbb{Z}^D} e^{r(|\Lambda|-1)}\|h_\Lambda\|_\infty < \infty \qquad r > 0$$

- Specific energy of TI state

$$E_h(\rho) = \lim_{\Lambda\uparrow\mathbb{Z}^D} \frac{\rho(H_\Lambda^h)}{|\Lambda|}$$

Gibbs states

- Specific entropy of TI state

$$s(\rho) = \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{S(\rho_\Lambda)}{|\Lambda|}$$

- Equilibrium states of TI interaction: Maximizers of

$$s(\rho) - E_h(\rho)$$

- Always exist but in general are not unique
- Satisfy KMS condition
- Local Gibbs states (NOT equal to marginals of equilibrium states)

$$\omega_\Lambda^h = \frac{e^{-H_\Lambda^h}}{\mathrm{Tr}\, e^{-H_\Lambda^h}} \in \mathcal{S}_\Lambda$$

- ρ is equilibrium state iff

$$\lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{S(\rho_\Lambda \| \omega_\Lambda^h)}{|\Lambda|} = 0$$

The specific quantum W_1 distance

- Specific W_1 distance for TI quantum states

$$w_1(\rho,\sigma) = \lim_{\Lambda\uparrow\mathbb{Z}^D} \frac{\|\rho_\Lambda - \sigma_\Lambda\|_{W_1}}{|\Lambda|}$$

- Lipschitz constant for TI quantum interactions

$$\|h\|_L = \partial_0 \sum_{0\in\Lambda\subset\subset\mathbb{Z}^D} h_\Lambda$$

- Duality

$$w_1(\rho,\sigma) = \sup_{\|h\|_L\leq 1} \left(E_h(\rho) - E_h(\sigma)\right)$$

- Continuity of the specific entropy

$$|s(\rho) - s(\sigma)| \leq h_2(w_1(\rho,\sigma)) + w_1(\rho,\sigma)\ln(d^2-1)$$

w_1-Gibbs states

- TI state ρ is w_1-Gibbs state of TI interaction h if

$$\lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{\left\| \rho_\Lambda - \omega_\Lambda^h \right\|_{W_1}}{|\Lambda|} = 0$$

- If it exists, w_1-Gibbs state is unique and is equilibrium state!
- TI interaction h satisfies TCI with constant c if for any TI state ρ

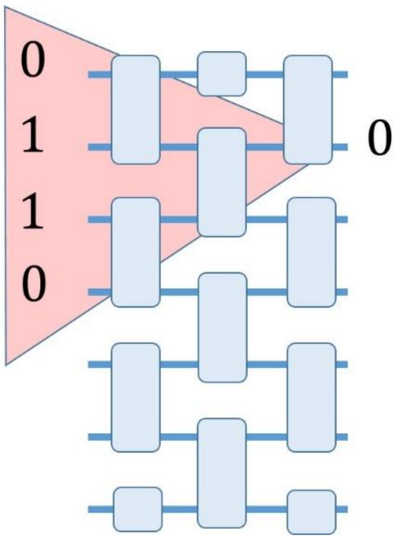
$$\limsup_{\Lambda \uparrow \mathbb{Z}^D} \frac{\|\rho_\Lambda - \omega_\Lambda^h\|_{W_1}^2}{|\Lambda|^2} \le \frac{c}{2} \lim_{\Lambda \uparrow \mathbb{Z}^D} \frac{S(\rho_\Lambda \| \omega_\Lambda^h)}{|\Lambda|}$$

- In this case, h has unique equilibrium state which is also w_1-Gibbs state
- TCI satisfied above critical temperature by any finite-range commuting interaction

Shallow quantum circuits

- Expand W_1 distance by at most twice the size of the largest light-cone of a qudit

$$\left\| U\,\rho\,U^{\dagger} - U\,\sigma\,U^{\dagger} \right\|_{W_1} \leq 2B(U)\,\|\rho - \sigma\|_{W_1}$$



Quadratic concentration for product states

- ω product state

$$\mathrm{Var}_\omega H \leq n \, \|H\|_L^2$$

- ρ output of quantum circuit with blow-up *B*

$$\mathrm{Var}_\rho H \leq 4n \, B^2 \, \|H\|_L^2$$

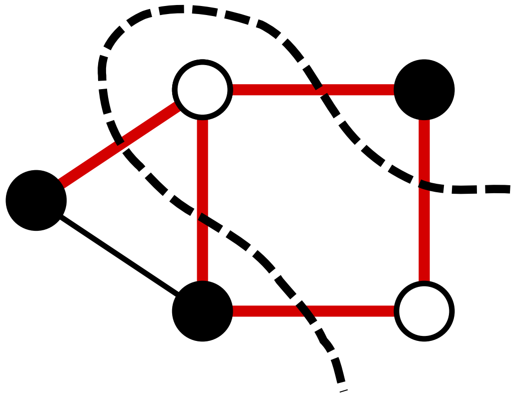
- See [Anshu, Metger, arXiv:2209.02715] for Gaussian concentration of observables diagonal in computational basis

Combinatorial optimization

- Goal: find bit string that maximizes cost function *C*
- Local cost: sum of functions each depending on *O*(1) bits
- Efficient classical algorithms usually achieve

$$C = a\,C_{\max} \qquad 0 < a \leq 1$$

- **Example:** maximum cut problem, i.e., find the bipartition of a graph that maximizes the # of edges connecting the two parts
- Associate one bit to each vertex, set to 1 bits in second half of bipartition
- NP complete!



Variational quantum algorithms

- Associate one qubit to each bit, quantum Hamiltonian to cost function

$$H = \sum_{x \in \{0,1\}^n} C(x) \, |x\rangle\langle x|$$

- Train parametric quantum circuit to generate high-energy states
- **Example:** Quantum Approximate Optimization Algorithm (QAOA)
- Alternate time evolution with *H* and mixing Hamiltonian

$$\left(\prod_{k=1}^{P} e^{-i\gamma_k \sum_{i=1}^{n} X_i} \, e^{-i\beta_k H} \right) |+\rangle^{\otimes n}$$

Limitations of QAOA for MaxCut

- Toy model: *D*-regular bipartite graph (maxcut = *n D / 2*)
- Technical assumption:

$$C(x) \geq \left(\frac{D}{2} - \sqrt{D-1}\right) \min\{|x|, n-|x|\} \qquad \forall x \in \{0,1\}^n$$

- Satisfied by Ramanujan expander graphs with $D \geq 3$ and for large *n* by random *D*-regular graphs with high probability
- Observation [Bravyi *et al.*, PRL 125, 260505 (2020)]: QAOA circuit commutes with $X^{\otimes n}$
- Probability distribution of output measurement symmetric wrt flipping all bits and cannot be concentrated on single string

Limitations of QAOA for MaxCut

- Result: if

$$\mathrm{Tr}\left[\rho\, H\right] \geq C_{\max}\left(\frac{5}{6}+\frac{\sqrt{D-1}}{3D}\right)$$

then the quadratic concentration inequality implies

$$P \geq \frac{1}{2\log\left(D+1\right)}\log\frac{n}{576} = \Omega(\log n)$$

- Holds for any circuit and initial state commuting with $X^{\otimes n}$

- Improves Bravyi *et al.*

$$P \geq \frac{1}{3\left(D+1\right)}\log_2\frac{n}{4096}$$

Further applications

- Quantum Wasserstein Generative Adversarial Networks

[Kiani, GdP, Marvian, Liu, Lloyd, Quantum Sci Technol 7, 045002 (2022)]

- Design of quantum error correcting codes

[Zoratti, GdP, Kiani, Nguyen, Marvian, Lloyd, Giovannetti, Phys. Rev. A 108, 022611 (2023)]

- Efficient learning of quantum states

[Rouzé, França, arXiv:2107.03333]

[Onorati, Rouzé, França, Watson, arXiv:2301.12946]

[GdP, Klein, Pastorello, arXiv:2309.08426]

- Quantum rate-distortion theory