Quantum spin systems

Def A qudit is a quantum system with Hilbert space $\mathbb{C}^d$.

A system of n qudits has Hilbert space $\mathcal{H}_n = \left(\mathbb{C}^d\right)^{\otimes n}$.

Let $\mathcal{O}_n = \mathcal{O}\left(\left(\mathbb{C}^d\right)^{\otimes n}\right)$, $\mathcal{S}_n = \mathcal{S}\left(\left(\mathbb{C}^d\right)^{\otimes n}\right)$

Let $\{|1\rangle \ldots |d\rangle\}$ be the canonical basis of $\mathbb{C}^d$

$\forall x \in \{1 \ldots d\}^n$, let $|x\rangle = |x_1\rangle \otimes \ldots \otimes |x_d\rangle$ s.t.

$\{|x\rangle : x \in \{1 \ldots d\}^n\}$ is an orthonormal basis of $\mathcal{H}_n$.

We associate to any probability distribution p on $\{1 \ldots d\}^n$ the state of n qudits

$$\rho_p = \sum_{x \in \{1 \ldots d\}^n} p(x)\, |x\rangle\langle x|.$$

We associate to any function $f : \{1 \ldots d\}^n \to \mathbb{R}$ the observable

$$O_f = \sum_{x \in \{1 \ldots d\}^n} f(x)\, |x\rangle\langle x|.$$

Def The Hamming distance between $x, y \in \{1 \ldots d\}^n$ is

$$h(x,y) = \left|\{ i \in \{1 \ldots d\} : x_i \neq y_i \}\right|$$

Quantum optimal transport

Classical OT: $(X\ d)$ metric space, find distance W on the set of the probability measures on X s.t. $W(\delta_x, \delta_y) = d(x,y) \ \forall x,y \in X$

In the quantum setting we have an obvious counterpart of d (at least not on all pure states).

Several proposals based on quantum generalizations of:

1) Dynamical formulation (Carlen Maas). Hard to compute; recovery of the classical case?

2) Couplings

- as probability measures (Golse ...) Hard to enforce faithfulness
  No triangle inequality
- as stochastic maps (DP Trevisan) Recovers classical case only in semiclassical limit

3) Lipshitz constant

- from differential structure (Dalla Rozé) No recovery of classical case
- from tensor product structure (DP ...) ⟵

4) Convert quantum state into classical probability distribution with measurements (Zyczkowski) Recovers classical case only in semiclassical limit

The quantum Lipshitz constant

Goal: obtain a quantum generalization of the Hamming distance and of the W_1 distance on the set of the probability distributions on $\{1 \dots d\}^n$ endowed with the Hamming distance

We recall that $\|p-q\|_{W_1} = \max\left\{\mathbb{E}_p f - \mathbb{E}_q f : \|f\|_L \le 1\right\}$

Let us find a quantum generalization of the Lipshitz constant

Given $f : \{1 \dots d\}^n \to \mathbb{R}$, we have

$$\|f\|_L = \max_{x, y \in \{1 \dots d\}^n} \frac{|f(y) - f(x)|}{h(y, x)} .$$

This definition cannot be directly generalized to quantum observables.

Given $x, y \in \{1 \dots d\}^n$, let $h = h(x, y)$. Then, $\exists\, x^{(0)} = x, x^{(1)}, \dots, x^{(h)} = y$ s.t. $h(x^{(i)}, x^{(i+1)}) = 1$. Therefore,

$$\|f\|_L = \max\left\{|f(y) - f(x)| : x, y \in \{1 \dots d\}^n, h(x, y) = 1\right\} .$$

$\forall\, i = 1 \dots n$, let

$$\partial_i f = \max\left\{|f(y) - f(x)| : x, y \in \{1 \dots d\}^n, x_j = y_j \forall j \neq i\right\}$$

n.t. $\|f\|_L = \max_{i=1..n} \partial_i f$.

Let us fix $i=n$. We have

$$\partial_n f = \max_{x\in\{1..d\}^{n-1}} \max_{x_n, y_n \in\{1..d\}} |f(x x_n) - f(x y_n)|$$
$$= 2 \max_{x\in\{1..d\}^{n-1}} \min_{g(x)\in\mathbb{R}} \max_{x_n\in\{1..d\}} |f(x x_n) - g(x)|$$
$$= 2 \min\{\|f-g\|_\infty : g:\{1..d\}^{n-1}\to\mathbb{R}\}$$

(Set $g(x) = \frac{1}{2}\left(\max_{x_n\in\{1..d\}} f(x x_n) - \min_{x_n\in\{1..d\}} f(x x_n)\right)$)

We can generalize this definition to the quantum setting!

<u>Def</u> $\forall X \in \mathcal{O}_n$, $i=1..n$ we define

$$\partial_i X = 2 \min\{\|X-Y\|_\infty : Y\in\mathcal{O}_n \text{ does not act on qudit } i\}$$

$$\|X\|_L = \max_{i=1..n} \partial_i X$$

<u>Obs</u> $\forall f:\{1..d\}^n\to\mathbb{R}$ we have $\partial_i \mathcal{O}_f = \partial_i f$ and $\|\mathcal{O}_f\|_L = \|f\|_L$.

The quantum W_1 distance

Def Let $\mathcal{O}_n^T = \{X \in \mathcal{O}_n : \operatorname{Tr} X = 0\}$.

Obs $\forall \rho, \sigma \in \mathcal{S}_n$ we have $\rho - \sigma \in \mathcal{O}_n^T$

Def (W_1 norm) Let $X \in \mathcal{O}_n^T$. We define

$$\|X\|_{W_1} = \max\{\operatorname{Tr}[XY] : Y \in \mathcal{O}_n, \|Y\|_L \le 1\}.$$

Thm Let $X \in \mathcal{O}_n^T$. Then,

$$\|X\|_{W_1} = \frac{1}{2}\min\left\{\sum_{i=1}^n \|X^{(i)}\|_1 : X^{(i)} \in \mathcal{O}_n^T,\ \operatorname{Tr}_i X^{(i)} = 0,\ X = \sum_{i=1}^n X^{(i)}\right\} \quad (\#)$$

Prop $\forall X \in \mathcal{O}_n^T$ we have $\frac{1}{2}\|X\|_1 \le \|X\|_{W_1} \le \frac{n}{2}\|X\|_1$.

Moreover, if $\exists i \in \{1..n\}$ s.t. $\operatorname{Tr}_i X = 0$, and in particular if $n=1$ we have

$$\|X\|_{W_1} = \frac{1}{2}\|X\|_1.$$

Proof Let $X \in \mathcal{O}_n^T$. For any $X^{(1)} .. X^{(n)}$ as in $(\#)$ we have

$\frac{1}{2}\|X\|_1 \le \frac{1}{2}\sum_{i=1}^n \|X^{(i)}\|_1$, therefore $\frac{1}{2}\|X\|_1 \le \|X\|_{W_1}$.

Let $X = X_+ - X_-$ where $X_\pm \ge 0$ and $\operatorname{Im} X_+ \perp \operatorname{Im} X_-$.

Let $t = \frac{1}{2}\|X\|_1 = \operatorname{Tr} X_+ = \operatorname{Tr} X_-$, and let $\rho^+ = \frac{X_+}{t}$, $\rho^- = \frac{X_-}{t}$

We have $X = t(\rho^+ - \rho^-) = t\sum_{i=1}^{m}\left(\rho^-_{1\ldots i-1}\otimes\rho^+_{i\ldots m} - \rho^-_{1\ldots i}\otimes\rho^+_{i+1\ldots m}\right)$

Let $X^{(i)} = t\left(\rho^-_{1\ldots i-1}\otimes\rho^+_{i\ldots m} - \rho^-_{1\ldots i}\otimes\rho^+_{i+1\ldots m}\right)$ s.t.

$\mathrm{Tr}_i X^{(i)} = 0$. Then, $\|X\|_{W_1} \leq \frac{1}{2}\sum_{i=1}^{m}\|X^{(i)}\|_1 \leq \frac{1}{2}\cdot m\cdot t\cdot 2$

$= \frac{m}{2}\|X\|_1$.

Let $\mathrm{Tr}_i X = 0$. We have proved that $\frac{1}{2}\|X\|_1 \leq \|X\|_{W_1}$.

Choosing $X^{(i)} = X$, $X^{(j)} = 0$ $\forall j\neq i$ we get $\|X\|_{W_1} \leq \frac{1}{2}\|X\|_1$.

Def $\rho,\sigma \in \mathcal{S}_m$ are neighboring if $\exists\, i\in\{1\ldots m\}$ s.t. $\mathrm{Tr}_i\rho = \mathrm{Tr}_i\sigma$

i.e. if they differ in only one qubit

Prop The unit ball of the W_1 norm is the convex hull of the differences between neighboring states.

Proof 1) Let $\rho,\sigma\in\mathcal{S}_m$ s.t. $\mathrm{Tr}_i\rho = \mathrm{Tr}_i\sigma$. Then, $\|\rho-\sigma\|_{W_1} =$

$= \frac{1}{2}\|\rho-\sigma\|_1 \leq 1$ and $\rho-\sigma\in$ unit ball of W_1.

2) Let $X\in\mathcal{O}_m^T$ s.t. $\|X\|_{W_1}\leq 1$. Then, $\exists\, X^{(1)}\ldots X^{(m)}\in\mathcal{O}_m^T$: $\mathrm{Tr}_i X^{(i)} = 0$

$X = \sum_{i=1}^{m} X^{(i)}$, $\frac{1}{2}\sum_{i=1}^{m}\|X^{(i)}\|_1 \leq 1$. For any $i=1\ldots m$, let $X^{(i)} = X^{(i)}_+ - X^{(i)}_-$

with $X_\pm^{(i)} \geq 0$ s.t. $\mathrm{Im}\, X_+^{(i)} \perp \mathrm{Im}\, X_-^{(i)}$. Let $p_i = \mathrm{Tr}\, X_+^{(i)} = \mathrm{Tr}\, X_-^{(i)} = \frac{1}{2}\|X^{(i)}\|_1$

Let $\rho^{(i)} = \frac{X_+^{(i)}}{p_i}$, $\sigma^{(i)} = \frac{X_-^{(i)}}{p_i} \in \mathcal{S}_n$, s.t. $\mathrm{Tr}_i(\rho^{(i)} - \sigma^{(i)}) =$

$= \frac{1}{p_i} \mathrm{Tr}_i\, X^{(i)} = 0$. We have $X = \sum_{i=1}^{n} p_i (\rho^{(i)} - \sigma^{(i)})$, $p_i \geq 0$,

$\sum_{i=1}^{n} p_i = \frac{1}{2} \sum_{i=1}^{n} \|X^{(i)}\|_1 \leq 1$, therefore $X \in$ convex hull.

Prop Let p, q be probability distributions on $\{1 .. d\}^n$. Then,

$\|\rho_p - \rho_q\|_{W_1} = \|p - q\|_{W_1}$. In particular, $\forall x, y \in \{1 .. d\}^n$ we have

$\| |x\rangle\langle x| - |y\rangle\langle y| \|_{W_1} = h(x, y)$

Prop The quantum W_1 norm is invariant with respect to permutations of the qudits and unitary operations acting on one qudit, and non-increasing with respect to quantum channels acting on one qudit

Prop Let $\{1 .. n\} = A \cup B$ with $A \cap B = \phi$. Then, $\forall \rho, \sigma \in \mathcal{S}_n$ we have $\|\rho - \sigma\|_{W_1} \geq \|\rho_A - \sigma_A\|_{W_1} + \|\rho_B - \sigma_B\|_{W_1}$ with equality if $\rho = \rho_A \otimes \rho_B$, $\sigma = \sigma_A \otimes \sigma_B$.

Prop Let $A \subseteq \{1 .. n\}$ and let $\rho, \sigma \in \mathcal{S}_n$ s.t. $\mathrm{Tr}_A \rho = \mathrm{Tr}_A \sigma$.

Then, $\|\rho-\sigma\|_{W_1} \leq |A| \frac{d^2-1}{d^2} \|\rho-\sigma\|_1$.

Continuity of the von Neumann entropy

Def Let $\rho \in \mathcal{S}(\mathcal{H})$. Then, $S(\rho) = -\mathrm{Tr}[\rho \ln \rho]$.

Prop Let $\rho, \sigma \in \mathcal{S}_n$ be neighboring. Then, $|S(\rho) - S(\sigma)| \leq 2 \ln d$

This robustness cannot be captured by a continuity bound wrt the trace distance.

Thm Let $\rho, \sigma \in \mathcal{S}_n$ and let $w = \frac{1}{n}\|\rho-\sigma\|_{W_1}$. Then,

$$\frac{1}{n}|S(\rho) - S(\sigma)| \leq h_2(w) + w \ln(d^2-1).$$

Quantum Marton's inequality

Def $\forall \rho, \sigma \in \mathcal{S}(\mathcal{H})$, let $S(\rho\|\sigma) = \mathrm{Tr}[\rho(\ln\rho - \ln\sigma)]$.

Thm (Pinsker's inequality) $\forall \rho, \sigma \in \mathcal{S}(\mathcal{H})$ we have

$$\frac{1}{2}\|\rho-\sigma\|_1 \leq \sqrt{\frac{1}{2} S(\rho\|\sigma)}$$

Pinsker's inequality does not have a good behavior wrt the tensor product:

$$\frac{1}{2}\|\rho^{\otimes n} - \sigma^{\otimes n}\|_1 \leq 1, \text{ while } \sqrt{\frac{1}{2} S(\rho^{\otimes n}\|\sigma^{\otimes n})} = \sqrt{\frac{n}{2} S(\rho\|\sigma)} = \Omega(\sqrt{n})$$

Thm Let $\omega \in \mathcal{S}_n$ be a product state, i.e., $\omega = \omega_1 \otimes \dots \otimes \omega_n$, $\omega_i \in \mathcal{S}_1$.

Then, $\forall \rho \in \mathcal{S}_n$ we have $\frac{1}{n}\|\rho - \omega\|_{W_1} \leq \sqrt{\frac{1}{2n} S(\rho\|\omega)}$.

Concentration inequalities

Talagrand's inequality: any smooth function of many variables is essentially constant. We generalize it to quantum observables, where the smoothness is quantified by the quantum Lipschitz constant

Thm Let $H \in \mathcal{O}_n$ with $\mathrm{Tr}\, H = 0$. Then,

$$\frac{1}{d^n}\mathrm{Tr}\, e^{H} \leq \exp \frac{n\|H\|_L^2}{8}$$

Cor Most of the eigenvalues of H are $O\left(\sqrt{n}\,\|H\|_L\right)$, i.e., $\forall \delta > 0$ we have

$$\frac{1}{d^n}\dim\left(H \geq \delta\sqrt{n}\,\|H\|_L\right) \leq e^{-2\delta^2}.$$

Does the same hold for any product state? Open problem!

Conj $\exists\, C > 0 : \forall n \in \mathbb{N}, \forall \omega \in \mathcal{S}_n$ product, $\forall H \in \mathcal{O}_n$ with $\mathrm{Tr}\, H = 0$ we have

$$\mathrm{Tr}\left[\omega\, e^{H}\right] \leq \exp\left(n\, C\, \|H\|_L^2\right)$$