Quantum Mean-Field Filtering and Control

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Motivation

The 21 century is seeing the emergence of the first truly quantum technologies; that is, technologies that rely on the counter-intuitive properties of individual quantum systems and can often outperform any conventional technology. Examples include quantum computing, which promises to be better than conventional computing for certain problems. To realize these promises, it's necessary to understand the measurement and control of quantum systems.

Formalism

In the theory of open quantum system undergoing continuous measurements, the state of the system of N-quantum particles is described by what we call a density matrix where the dynamics evolution is given by stochastic differential equation known as Belavkin Equation :

$$\mathrm{d}\rho_t^N = \sum_{k=1}^{N}\Big(\mathcal{H}_k[\rho_t^N] + \mathcal{L}_k[\rho_t^N]\Big)\mathrm{d}t + \mathcal{R}_k[\rho_t^N]\mathrm{d}W_t^{(k)}$$

- $\mathcal{H}_k(\rho) := -\mathrm{i}[u(\rho_t^{(k)})H_k + \sum_{j>k}\frac{A_{jk}}{N}, \rho]$
- $\mathcal{L}_k(\rho) := L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}$
- $\mathcal{R}(\rho) := L_k\rho + \rho L_k^\dagger - tr\big((L_k + L_k^\dagger)\rho\big)\rho.$
- $\mathcal{S} := \{\mathcal{M}_{d^N}(\mathbb{C}), \rho = \rho^\dagger \geq 0, tr(\rho) = 1\}$

The process $(\rho_t^N)_{t\geq 0}$ is valued on the set densities matrices $\mathcal{S}$.
Where H_k is hermitian matrix (i.e $H = H^\dagger$) called hamiltonian of control and u is scalar function, and the L_k are matrix where $L_k + L_k^\dagger$ represent the observable quantities for the $k-$th particle. The matrix A_{jk} denotes the pairwise interaction between each particles.
To describe the dynamics of (ρ_t^N) We need $d^{2N} - 1$ real numbers, so the complexity grows exponentially fast and deal with such equations becomes impossible.
However when N grows by symmetries of the pairwise interaction we can expect some averaging and decorrelation between particles. Typical behaviour of particle emmerge, this what we called mean-field limit in this setting we deal with Belavkin stochastic differential equation of McKean-Vlasov type :

$$\begin{aligned}\mathrm{d}\gamma_t &= (-\mathrm{i}[u(\gamma_t)H + A^{m_t}, \gamma_t])\mathrm{d}t \\ &+ \Big(L\gamma_t L^\dagger - \frac{1}{2}\{L^\dagger L, \gamma_t\}\Big)\,\mathrm{d}t \\ &+ \Big(\gamma_t L^\dagger + L\gamma_t - \mathrm{tr}\left((L + L^\dagger)\gamma_t\right)\gamma_t\Big)\mathrm{d}W_t,\end{aligned}$$

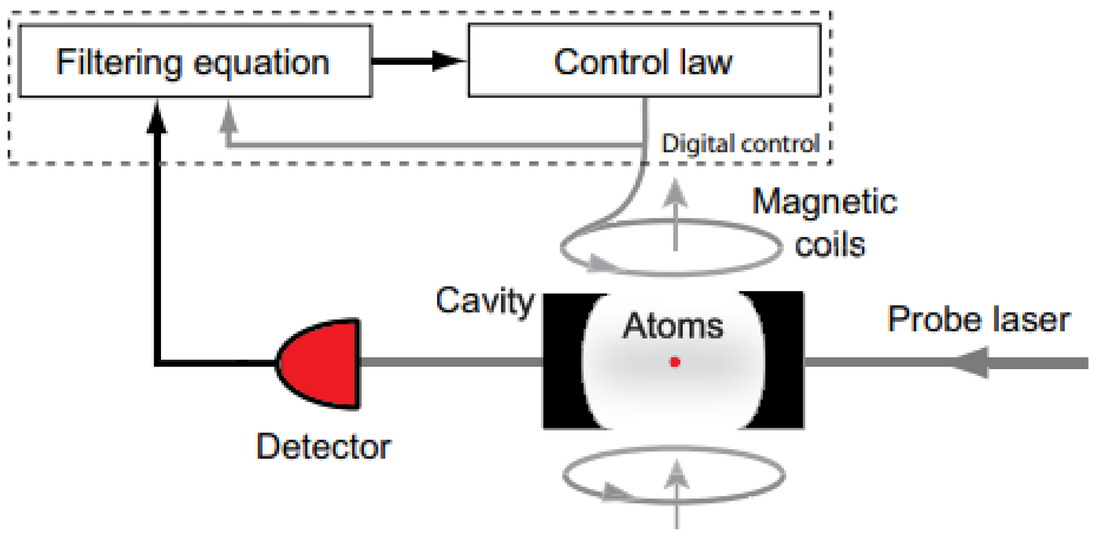
Where, $m_t = \mathbb{E}[\gamma_t]$. To justify the approximation, we have to show that ρ^N asymptotically becomes close to $\gamma^{\otimes N}$. To measure a deviation from ρ^N to $\gamma^{\otimes N}$, we consider the following quantity

$$\alpha_{N,j}(t) = 1 - tr\big(\gamma_t\rho_t^j\big) = 1 - tr\big(\gamma_t^j\rho_t^N\big),$$

Theorem 1 (Propagation of chaos)**.** *Let u be bounded and Lipschitz, then MF-Belavkin equation is well posed and valued in $\mathcal{S}_d$.Moreover we have propagation of chaos i.e $\exists c$ s.t*

$$\mathbb{E}\big[\alpha_N(t)\big] \leq e^{ct}\left(\alpha_N(0) + \frac{1}{\sqrt{N}}\right).$$

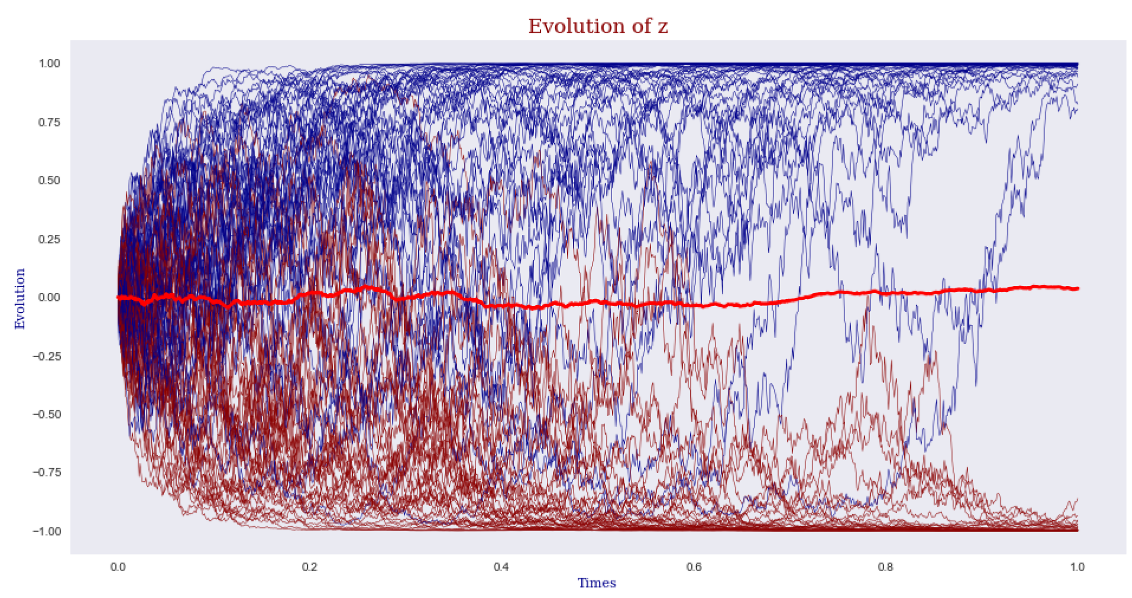
Synoptic of Quantum Feedback Control



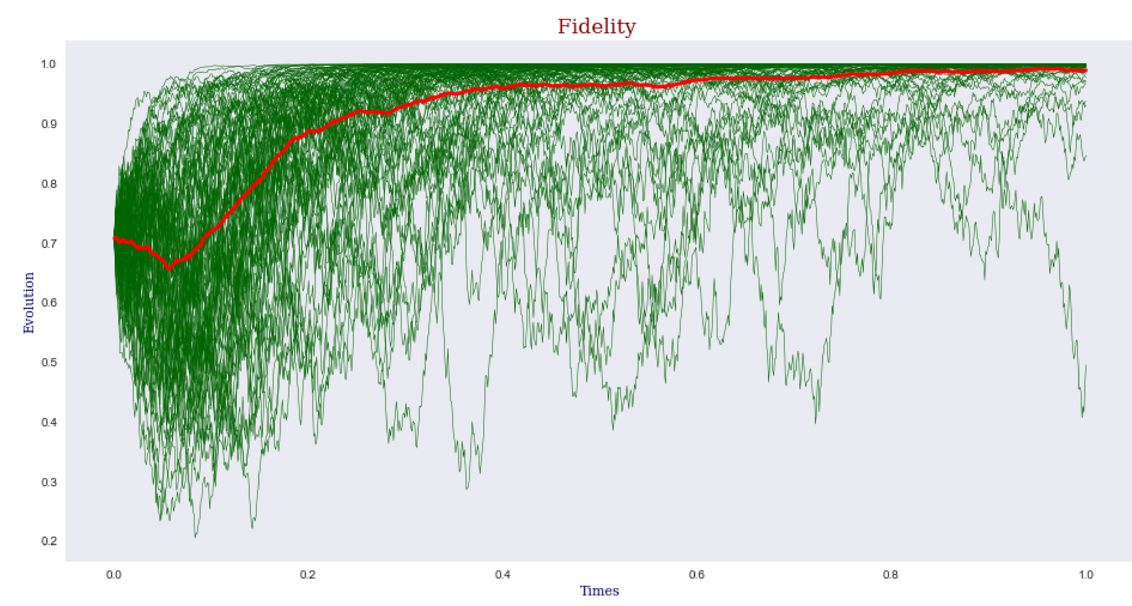
A typical feedback control scenario in quantum optics. A probe laser scatters off a cloud of atoms in an optical cavity, and is ultimately detected. The detected signal is processed by a controller which feeds back to the system through a time varying magnetic field.

Figure 1: Diagram Quantum Feedback Control.

Illustration through an example



Quantum states reduction



Stabilization with feedback

We consider the case of N-qubit system, interacting through a Hamiltonian of MF type.
$a(1, 1; 2, 2) = a(2, 2; 1, 1)$.
For each particle we an observation channel $L_j = {\sigma_z}^j$ and a controlled Hamiltonian $H_j = {\sigma_x}^j$. The evolution of the N-particles systems is given by the following equation:

$$\mathrm{d}\rho_t^N = -\mathrm{i}[H_t + \sum_{j=1}^{N}\Big({\sigma_z}^j\rho_t^N{\sigma_z}^j - \rho_t^N\Big)\mathrm{d}t + \sum_{j=1}^{N}\Big(\rho_t^N{\sigma_z}^j + {\sigma_z}^j\rho_t^N - 2tr\big({\sigma_z}^j\rho_t^N\big)\rho_t^N\Big)\mathrm{d}W_t^j.$$

Note that the simulation of $\boldsymbol{\rho}^N$ requires $4^N - 1$ real stochastic differential equations and the complexity is $O(4^N)$.
The MF Belavkin equation parametrize by reals numbers is represented as follows:

$$\begin{aligned}\mathrm{d}x_t &= \big(-y_t - x_t + \mathbb{E}[z_t]y_t\big)\mathrm{d}t - x_t z_t\mathrm{d}W_t \\ \mathrm{d}y_t &= \big(x_t - y_t + u(\gamma_t)z_t - x_t\mathbb{E}[z_t]\big)\mathrm{d}t + y_t z_t\mathrm{d}W_t \\ \mathrm{d}z_t &= -u(\gamma_t)x_t\mathrm{d}t + (1 - z_t^2)\mathrm{d}W_t\end{aligned}$$

To simulate the MF equation, we need to solve only three real stochastic differential equations. Nevertheless, we need to approximate $\mathbb{E}[x_t], \mathbb{E}[y_t], \mathbb{E}[z_t]$ using an N-particle system, which yields a complexity $O(N)$.

We start by studying the asymptotic behavior of our system when the feedback control is turned off, i.e., $(u \equiv 0)$. We observe a quantum state reduction property, i.e $(\gamma_t)_{t\geq 0}$ converges to one of the eigenstates of L, i.e., $\{\rho_e, \rho_g\}$ with

$$\rho_g := \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \ \rho_e := \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

that are the equilibrium points of the MF equation. Further, to ensure that the system attains a prescribed target, for example ρ_e, we adapt a feedback law control u given by
$u(\gamma) := -8\mathrm{i}\,\mathrm{tr}\left([\sigma_x, \gamma]\rho_e\right) + 5\big(1 - \mathrm{tr}(\gamma\rho_e)\big)$. the right illustration shows that the stabilization is achieved.